Binomial formula

Acquainted yourself first with a character : n ! $\longrightarrow$ where "!" is the factorial operator.

n!=n ∘ (n-1) ∘ (n-2) ∘ …∘3∘ 2 ∘1

Example:

5!=5∘4∘3 ∘2 ∘1= 120 or 7! = 7 ∘6 ∘ 5 ∘4 ∘3 ∘2 ∘1 = 5040

By definition, is 0! = 1

In tasks we often use to separate the factorial as a product of few members and new factorial.

For example:

(n+2)! = (n+2)(n+1)n(n-1) ∘…∘2 ∘1

(n+2)! = (n+2)(n+1)**n !** or

(n+2)! = (n+2)(n+1)n **(n-1)!** or ….

Example 1. Reduce the fraction:

$$\frac{(n-1)!}{(n-3)!}$$

Solution: $\frac{(n-1)!}{(n-3)!}=\frac{(n-1)(n-2)(n-3)!}{(n-3)!}$ =(n-1)(n-2)

Example 2. Solve the equation:

$$\frac{(2x)!}{(2x-3)!}=\frac{20x!}{(x-2)!}$$

Solution:

$$\frac{(2x)!}{(2x-3)!}=\frac{20x!}{(x-2)!}$$

$$\frac{(2x)(2x-1)(2x-2)(2x-3)!}{(2x-3)!}=\frac{20x(x-1)(x-2)!}{(x-2)!}$$

(2x)(2x-1)(2x-2)= 20x(x-1)

2x (2x-1)2(x-1)= 20 x(x-1)

2x-1 = 5 and from here **is** **x=3**

$\binom{n}{k}$-is interpreted as the number of k-element subsets of an n-element set, that is the number of ways that k things can be "chosen" from a set of n things.

$\binom{n}{k}$ is often read as **"*n* choose *k*"** and called the **choose function** of n and k.

$$\binom{n}{k} = \frac{n(n-1)(n-2)...(n-k+1)}{k!}$$

Examples: $\binom{10}{2} = \frac{10 \circ 9}{2 \circ 1} = 45$, $\binom{15}{3} = \frac{15 \circ 14 \circ 13}{3 \circ 2 \circ 1} = 455$

To have **the speed in the work** we have to remember that:

$\binom{n}{0} = 1$ For example : $\binom{5}{0} = 1$ $\binom{12}{0} = 1$ .....

$\binom{n}{n} = 1$ For example : $\binom{7}{7} = 1$ $\binom{100}{100} = 1$ .....

$\binom{n}{1} = \binom{n}{n-1} = n$ For example: $\binom{4}{1} = \binom{4}{3} = 4$ $\binom{50}{1} = \binom{50}{49} = 50$

And most important: $\binom{n}{k} = \binom{n}{n-k}$

For example,we get to decide $\binom{20}{18}$.With this policy we resolve:

$\binom{20}{18} = \binom{20}{2} = \frac{20 \circ 19}{2 \circ 1}$ = 190. **It is easier!**

Now we can see how it seems **binomial form:**

$$\mathbf{(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n}$$

This formula can be easily proved by the application of mathematical induction.

What is important to notice?

- The development has always n +1 members
- **a** begins with the degree **n,** and any member of the following decreases until it comes to **zero**, while **b** starts from **zero**, and each member of the next growing until come to the degree **n**

- $\binom{n}{0}$, $\binom{n}{1}$, $\binom{n}{2}$,..., $\binom{n}{n-1}$ and $\binom{n}{n}$ are the **binomial coefficients**, and for them we have one interesting thing:

Paskal triangle

$\mathbf{(a+b)^0 = 1}$ **coefficient 1** 1

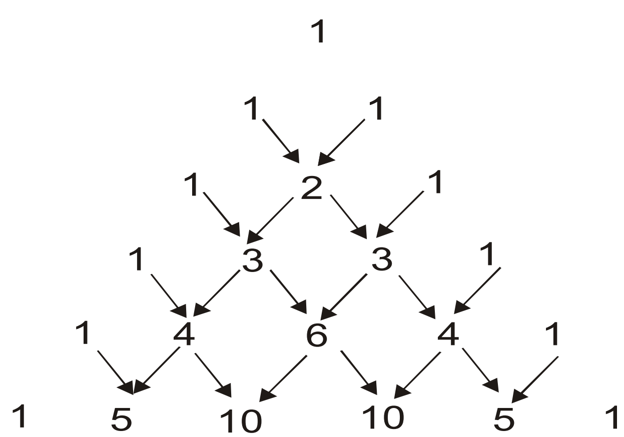
$\mathbf{(a+b)^1 = a+b}$ **coefficients are 1 and 1** 1 1

$\mathbf{(a+b)^2 = a^2+2ab+b^2}$ **coefficients are 1, 2, 1** 1 2 1

$\mathbf{(a+b)^3 = a^3+3a^2b+3ab^2+b^3}$ **coefficients are 1, 3, 3, 1** 1 3 3 1

$\mathbf{(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4}$ 1 4 6 4 1

coefficients are 1, 4, 6, 4, 1 etc..



.... And so on ....

General (any) member in the form of a developed stage is required by the formula:

$$\mathbf{T_{k+1} = \binom{n}{k} a^{n-k} b^k}$$

EXAMPLES:

1) $(3+2x)^5 = ?$

$(3+2x)^5 =$ [Here is $a=3$, $b=2x$ and $n=5$]

$$\binom{5}{0}3^5(2x)^o+\binom{5}{1}3^4(2x)^1+\binom{5}{2}3^3(2x)^2+\binom{5}{3}3^2(2x)^3+\binom{5}{4}3^1(2x)^4+\binom{5}{5}3^o(2x)^5$$

it's easier to extract the binomial coefficients,and fix them first:

$$\binom{5}{0}=\binom{5}{5}=1$$

$$\binom{5}{1}=\binom{5}{4}=5$$

$$\binom{5}{2}=\frac{5\cdot 4}{2\cdot 1}=10=\binom{5}{3}$$

$$=1\cdot 3^2\cdot 1+5\cdot 3^4\cdot 2\cdot x+10\cdot 3^3\cdot 2^2x^2+10\cdot 3^2\cdot 2^3\cdot x^3+5\cdot 3\cdot 2^4\cdot x^4+1\cdot 1\cdot 2^5\cdot x^5=$$
$$=243+810x+1080x^2+720x^3+240x^4+32x^5$$

2) $(1+i)^6 = ?$

$(1+i)^6 =$ [Here is $a=1$, $b=i$ and $n=6$]

$$\binom{6}{0}1^6\cdot i^o+\binom{6}{1}1^5\cdot i^1+\binom{6}{2}1^4\cdot i^2+\binom{6}{3}1^3\cdot i^3+\binom{6}{4}1^2\cdot i^4+\binom{6}{5}1^1\cdot i^5+\binom{6}{6}1^o\cdot i^6$$

$$\binom{6}{0}=\binom{6}{6}=1$$

$$\binom{6}{1}=\binom{6}{5}=6$$

$$\binom{6}{2}=\frac{6\cdot 5}{2\cdot 1}=15=\binom{6}{4}$$

$$\binom{6}{3}=\frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}=20$$

To remind you:

$$\left.\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}\right\} \text{ So: } \begin{aligned} i^5 &= i^4 \cdot i = i \\ i^6 &= i^4 \cdot i^2 = -1 \end{aligned}$$

Let's go back to the task:

$$\begin{aligned} &= 1 \cdot 1 \cdot 1 + 6 \cdot 1 \cdot i + 15 \cdot 1 \cdot (-1) + 20 \cdot 1 \cdot (-i) + 15 \cdot 1 \cdot 1 + 6 \cdot i + 1 \cdot 1(-1) \\ &= 1 + 6i \cdot 15 - 20i + 15 + 6i - 1 \\ &= -8i \end{aligned}$$

So: $(1+i)^6 = -8i$

3) Determine the **fifth member** in the form of a developed stage: $\left(x^{\frac{1}{2}} + x^{\frac{2}{3}}\right)^{12}$

Solution:

$$a = x^{\frac{1}{2}}, \quad b = x^{\frac{2}{3}}, \quad n = 12$$

We will use formula:

$$T_{k+1} = \binom{n}{k} a^{n-k} b^k$$

As the search is for the fifth member:

$$\begin{aligned} T_5 &= T_{4+1} \\ &= \binom{12}{4} \left(x^{\frac{1}{2}}\right)^{12-4} \left(x^{\frac{2}{3}}\right)^4 \\ &= \binom{12}{4} x^4 \cdot x^{\frac{8}{3}} \\ &= \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} x^{4+\frac{8}{3}} \\ &= 495 \cdot x^{\frac{20}{3}} \end{aligned}$$

4) Determine member that does not include x in developed form of $\left(x+x^{-2}\right)^{12}$

Solution:

$$a=x,\quad b=x^{-2},\quad n=12$$

We will use formula $T_{k+1}=\binom{n}{k}a^{n-k}b^{k}$ to find k

$$T_{K+1}=\binom{n}{k}a^{n-K}\cdot b^{K}$$
$$=\binom{12}{k}x^{12-k}\left(x^{-2}\right)^{k}$$
$$=\binom{12}{k}x^{12-k}\cdot x^{-2k}$$
$$=\binom{12}{k}x^{12-3k}$$

Since we need a member **that** does not contain **x,** carried out by comparison:

$$x^{12-3k}=x^{o}$$
$$12-3k=0$$
$$3k=12$$
$$k=4$$

So, in question was $\left(T_{4+1}=T_5\right)$ the fifth member. $T_{k+1}=\binom{12}{k}x^{12-3k}\rightarrow T_5=\binom{12}{4}$

5) The sum of coefficient of the first, second and third member in the development stage of $\left(x^2+\frac{1}{x}\right)^n$ is 46.

Find a member that does not contain x.

Solution:

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}=46$$
$$1+n+\frac{m(n-1)}{2}=46$$
$$2+2n+n^2-n=92$$
$$n^2+n-90=0$$
$$n_{1,2}=\frac{-1\pm 9}{2}$$
$$n=9$$

So: $a = x^2$, $b = \frac{1}{x}$, $n = 9$

$$T_{K+1} = \binom{n}{k} a^{n-K} \cdot b^K$$
$$= \binom{9}{k} (x^2)^{9-k} \left(\frac{1}{x}\right)^k$$
$$= \binom{12}{k} x^{18-k} \cdot x^k$$
$$= \binom{12}{k} x^{18-3k}$$

$$x^{18-3k} = x^o$$
$$18 - 3k = 0$$
$$3k = 18$$
$$k = 6$$

Means that there is a seventh member.

$$T_7 = \binom{9}{6} = \binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$$

6) Find the coefficients with x^3 in developed stage $\left(\frac{1}{4x} - 2x^2\right)^{12}$

Solution:

$$a = \frac{1}{4x}, b = -2x^2, n = 12$$

$$T_{K+1} = \binom{n}{k} a^{n-K} \cdot b^K$$
$$= \binom{12}{k} \left(\frac{1}{4x}\right)^{12-k} \cdot \left(-2x^2\right)^k$$
$$= \binom{12}{k} \left(\frac{1}{4}\right)^{12-k} x^{k-12} \cdot (-2)^k \cdot x^{2k}$$
$$= \binom{12}{k} \left(\frac{1}{4}\right)^{12-k} \cdot (-2)^k \cdot \underbrace{x^{3k-12}}_{x^3}$$

So: $x^{3k-12} = x^3$

$$3k - 12 = 3$$
$$3k = 15$$
$$k = 5$$

And the coefficients with x^3 will be:

$$\binom{12}{k} \left(\frac{1}{4}\right)^{12-k} (-2)^k =$$
$$\binom{12}{5} \left(\frac{1}{4}\right)^7 \cdot (-2)^5 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{1}{4^7} \cdot (-32)$$
$$= -\frac{99}{64} = -1{,}546875$$

7) We have $\binom{n}{1}:\binom{n}{2}=2:11$ and $\left(\frac{x}{\sqrt{4}}+\frac{\sqrt{y}}{x}\right)^n$. Determine the fifth member.

Solution:

$$\binom{n}{1}:\binom{n}{2}=2:11$$

$$n:\frac{n(n-1)}{2}=2:11$$

$$11n=n(n-1)$$

$$11n=n^2-n$$

$$n^2-12n=0$$

$$n(n-12)=0 \Rightarrow \underset{not\ a\ solution}{n=0} \quad \vee \quad n=12$$

Because $a=\frac{x}{\sqrt{4}}, b=\frac{\sqrt{y}}{x}, n=12$ and we must find the fifth member:

$$T_{K+1}=\binom{n}{k}a^{n-K}\cdot b^K$$

$$T_{K+1}=\binom{n}{k}a^{n-K}\cdot b^K$$

$$T_5=T_{4+1}=\binom{12}{4}\left(\frac{x}{\sqrt{y}}\right)^8\cdot\left(\frac{\sqrt{y}}{x}\right)^4$$

$$=\binom{12}{4}\frac{x^8}{y^4}\cdot\frac{y^2}{x^4}$$

$$=\frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1}\cdot\frac{x^4}{y^2}$$

$$=495x^4y^{-2}$$

8) At the railway station should be received in the same direction n people. On how many possible ways, considering the time of arrival, thay can arrive at the station?

Solution:

Think:

- Can all came at different time
- To arrive two together, the other in a different time
- To arrive three together, the other in a different time
- Etc.
- To arrive in groups 2
- To arrive in groups of 3
- itd - Etc.

Number of all the possibilities:

$$C_1^n + C_2^n + C_3^n + \ldots + C_n^n =$$
$$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} =$$

To calculate this, let us start from binomial formula :

$$(a+b)^n = \binom{n}{o} a^n b^o + \binom{n}{1} a^{n-1} b^1 + \ldots + \binom{n}{n} a^o b^n$$

If instead of **a** and **b** put **1**, you will receive:

$$(1+1)^n = \binom{n}{o} \cdot 1 \cdot 1 + \binom{n}{1} \cdot 1 \cdot 1 + \ldots + \binom{n}{n} \cdot 1 \cdot 1$$
$$2^n = \binom{n}{o} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

$$\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n - \binom{n}{o}$$
$$\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n - 1$$

Number of possibilities : $2^n - 1$