SECOND ORDER DIFFERENTIAL EQUATIONS (examples)

1. Solve the differential equation: $y`` = x + \sin x$

Solution:

$y`` = x + \sin x$ replacement $y` = p \longrightarrow y`` = p`$

$p` = x + \sin x$

$\frac{dp}{dx} = x + \sin x$

$dp = (x + \sin x)dx$ this is differential equation that separates the variable

$\int dp = \int (x + \sin x)dx$

$p = \frac{x^2}{2} - \cos x + c_1$ we've added a constant c_1 because we have decided one integral

$y` = \frac{x^2}{2} - \cos x + c_1$

$\frac{dy}{dx} = \frac{x^2}{2} - \cos x + c_1$

$dy = (\frac{x^2}{2} - \cos x + c_1)dx$

$\int dy = \int (\frac{1}{2}x^2 - \cos x + c_1)dx$

$y = \frac{1}{2}\frac{x^3}{3} - \sin x + c_1 x + c_2$ add the constant c_2

$y = \frac{x^3}{6} - \sin x + c_1 x + c_2$ this is a general integral (solution)

2. Find general integral for equation: $y`` + 2yy`^3 = 0$

Solution:

$y`` + 2yy`^3 = 0$ replacement: $y` = p$, then $y`` = p`p$ (see theoretical note)

$p`p + 2y\,p^3 = 0$

$p(p` + 2yp^2) = 0$ here is $p = 0$ or $p` + 2yp^2 = 0$

For p = 0 , we immediately get a solution: $y' = 0 \longrightarrow y = c$ (constant)

$p` + 2yp^2 = 0$

$$\frac{dp}{dy} = -2yp^2$$

$$\frac{dp}{p^2} = -2ydy$$ "separates the variable"

$$\int \frac{dp}{p^2} = \int -2ydy$$

$$-\frac{1}{p} = -2\frac{y^2}{2} + c_1$$

$$\frac{1}{p} = y^2 - c_1$$

$$p = \frac{1}{y^2 - c_1}$$ $y` = p$

$$y` = \frac{1}{y^2 - c_1}$$

$$\frac{dy}{dx} = \frac{1}{y^2 - c_1}$$

$(y^2 - c_1)dy = dx$

$$\frac{y^3}{3} - c_1 y = x + c_2$$ general integral

Therefore, solutions are: $y = c$ and $\frac{y^3}{3} - c_1 y = x + c_2$

3. Find general integral for equations:

a) $y`` - 3y` + 2y = 0$

b) $y`` - 2y` + y = 0$

c) $y`` - 2y` + 2y = 0$

d) $y^{IV} - 5y`` + 4y = 0$

Solution:

First,we solve the characteristic equation (see theory).

Take heed: In characteristic equation we can use replacement p or r or λ or same other letter!

You do the same as your professor!

a) $y'' - 3y' + 2y = 0$

$p^2 - 3p + 2 = 0$

$$p_{1,2} = \frac{3 \pm 1}{2} \Rightarrow p_1 = 2, p_2 = 1$$

$$y = c_1 e^{2x} + c_2 e^x$$

b) $y'' - 2y + y = 0$

$p^2 - 2p + 1 = 0$ solve the characteristic equation

$$p_{1,2} = \frac{2 \pm 0}{2} \Rightarrow p_1 = 1, p_2 = 1$$

$$y = c_1 e^x + c_2 x e^x$$

c) $y'' - 2y' + 2y = 0$

$p^2 - 2p + 2 = 0$

$$p_{1,2} = \frac{2 \pm 2i}{2} = \frac{2(1 \pm i)}{2} \Rightarrow p_1 = 1 + i, p_2 = 1 - i$$

$y = c_1 e^x \cos x + c_2 e^x \sin x$ (Again: see theoretical notes)

d) $y^{IV} - 5y'' + 4y = 0$

$p^4 - 5p^2 + 4 = 0$ replacement: $p^2 = t$

$t^2 - 5t + 4 = 0$

$t_{1,2} = \frac{5 \pm 3}{2} \Rightarrow t_1 = 4, t_2 = 1$ return replacement $p^2 = t$

$p^2 = 4$ or $p^2 = 1$, so: $p_1 = 2$, $p_2 = -2$, $p_3 = 1$, $p_4 = -1$

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$$

4. Find a general integral for equation: $y'' - y = x^2 + 1$

Solution:

First, find the solution of homogeneous equation:

$$y'' - y = 0$$

$$p^2 - 1 = 0$$

$$p_{1,2} = \frac{0 \pm 2}{2} \Rightarrow p_1 = 1, p_2 = -1$$

$$y_H = c_1 e^x + c_2 e^{-x}$$

We will use the method of undetermined coefficients in this case.

$\mathbf{Y = ax^2 + bx + c}$ where a, b, c are required coefficients

$Y' = 2ax + b$

$Y'' = 2a$

This change in: $y'' - y = x^2 + 1$

$$2a - (ax^2 + bx + c) = x^2 + 1$$

$$2a - ax^2 - bx - c = x^2 + 1$$

$$-ax^2 - bx + (2a - c) = x^2 + 1$$

Now we compare coefficients, with x^2, then with x, then “free members”

$$\left.\begin{array}{l} -a = 1 \\ -b = 0 \\ 2a - c = 1 \end{array}\right\}$$

From here is a = -1, b = 0 and c = -3 ⟶ $\mathbf{Y = ax^2 + bx + c}$

So:

$Y = -x^2 - 3$

$y = y_H + Y$ the final solution will be : $y = c_1 e^x + c_2 e^{-x} - x^2 - 3$

5. Solve the differential equation: $y``-y`=\frac{1}{1+e^x}$

Solution:

Find a solution corresponding homogeneous equation.

$y``-y`=0$

$p^2 - p = 0$

$p_{1,2}=\frac{1\pm 1}{2} \Rightarrow p_1=0, p_2=1$

$y_H = c_1 + c_2 e^x$

Here we will use method variation of parameters:

$c_1=c_1(x)$ and $c_2=c_2(x)$ post system:

$$\left.\begin{aligned} c`_1 + c`_2 e^x &= 0 \\ 0c`_1 + c`_2 e^x &= \frac{1}{1+e^x} \end{aligned}\right\} \text{ from second equation express } c_2$$

$c_2` = \frac{1}{e^x(1+e^x)}$

$c_2 = \int \frac{1}{e^x(1+e^x)}dx$ **In this integral we (as a trick) add up and down** e^x

$\int \frac{1}{e^x(1+e^x)}dx = \int \frac{1e^x}{e^x e^x(1+e^x)}dx = \int \frac{e^x}{e^{2x}(1+e^x)}dx$ replacement $\left| \begin{array}{c} e^x = t \\ e^x dx = dt \end{array} \right|$, then

$\int \frac{e^x}{e^{2x}(1+e^x)}dx = \int \frac{dt}{t^2(1+t)}$ this integral we solve as a integral of rational function!

$$\frac{1}{t^2(1+t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{(1+t)}$$

$1 = At(t+1) + B(t+1) + C t^2$

$1 = At^2 + At + Bt + B + C t^2$ group members

$1 = t^2(A+C) + t(A+B) + B$ compare coefficients, with t^2, then with t, then "free members"

A+C = 0
A+B = 0
B = 1 wrom here is A= - 1 i C = 1

Go back:

$$\frac{1}{t^2(1+t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{(1+t)} = \frac{-1}{t} + \frac{1}{t^2} + \frac{1}{(1+t)}$$

$$\int \frac{dt}{t^2(1+t)} = \int \left(\frac{-1}{t} + \frac{1}{t^2} + \frac{1}{(1+t)}\right)dt = -\ln|t| - \frac{1}{t} + \ln|t+1|$$ return replacement : $t = e^x$

$$= -\ln e^x - \frac{1}{e^x} + \ln(e^x+1)$$

$$= -x - \frac{1}{e^x} + \ln(e^x+1) + d_2$$ (d_2 is a constant)

$$c_2(x) = -x - \frac{1}{e^x} + \ln(e^x+1) + d_2$$

next we are looking for $c_1 = c_1(x)$

$$c`_1 + c`_2 e^x = 0$$

$$c`_1 = -c`_2 e^x = -\frac{1}{e^x(1+e^x)}e^x = -\frac{1}{1+e^x}$$

$$c_1 = \int -\frac{1}{1+e^x}dx = -\int \frac{1}{1+e^x}dx$$

Similar…

$c_1(x) = \ln(1+e^x) - x + d_1$

So, we get:

$$\left.\begin{aligned} c_1(x) &= \ln(1+e^x) - x + d_1 \\ c_2(x) &= -x - \frac{1}{e^x} + \ln(e^x+1) + d_2 \end{aligned}\right\}$$ This return in homogeneous solution:

$$y_H = c_1 + c_2 e^x$$

$y = \ln(1+e^x) - x + d_1 + (-x - \frac{1}{e^x} + \ln(e^x+1) + d_2)e^x$

$y = \ln(1+e^x) - x + d_1 - xe^x - 1 + e^x \ln(e^x+1) + d_2 e^x$

$y = d_1 + d_2 e^x - x - xe^x - 1 + \ln(1+e^x) + e^x \ln(e^x+1)$ this is a general solution (general integral)

6. Solve the differential equation: $y`` - 2y` + 2y = e^x \sin x$

Solution:

$y`` - 2y` + 2y = 0$

$p^2 - 2p + 2 = 0$

$$p_{1,2} = \frac{2 \pm 2i}{2} = \frac{2(1 \pm i)}{2} \Rightarrow p_1 = 1+i, p_2 = 1-i$$

$$y_H = c_1 e^x \cos x + c_2 e^x \sin x$$

$c_1 = c_1(x)$ and $c_2 = c_2(x)$

$$c`_1 e^x \cos x + c`_2 e^x \sin x = 0$$
$$c`_1 (e^x \cos x - e^x \sin x) + c`_2 (e^x \sin x + e^x \cos x) = \sin x$$

$$c`_1 \cos x + c`_2 \sin x = 0$$
$$c`_1 (\cos x - \sin x) + c`_2 (\sin x + \cos x) = \sin x$$

all divide with e^x

$c`_2 = -c`_1 \frac{\cos x}{\sin x}$ We expressed $c`_2$ and that we will replace in other equation

$$c`_1 (\cos x - \sin x) - c`_1 \frac{\cos x}{\sin x} (\sin x + \cos x) = \sin x$$

$$c`_1 (\cos x - \sin x) - c`_1 (\cos x + \frac{\cos^2 x}{\sin x}) = \sin x$$

$$c`_1 (\cos x - \sin x - \cos x - \frac{\cos^2 x}{\sin x}) = \sin x$$

$c`_1 (-\sin x - \frac{\cos^2 x}{\sin x}) = \sin x$ all multiply with (- sin x)

$c`_1 (\sin^2 x + \cos^2 x) = -\sin^2 x$ We know that: $\sin^2 x + \cos^2 x = 1$

$c`_1 = -\sin^2 x$ **use:** $\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2}(1 - \cos 2x)$

$$c_1 = -\int \frac{1}{2}(1-\cos 2x)dx = -\frac{1}{2}(x-\frac{1}{2}\sin 2x)+d_1$$

$$c_1 = -\frac{1}{2}x+\frac{1}{4}\sin 2x + d_1$$ c_2=?

$c`_2 = -c`_1 \frac{\cos x}{\sin x}$ and $c`_1 = -\sin^2 x$ then must be: $c`_2 = \sin^2 x \frac{\cos x}{\sin x} = \sin x \cos x$

$c`_2 = \sin x \cos x$ now integral

c_2= $\int \sin x \cos x dx = \left| \begin{matrix} \sin x = t \\ \cos x dx = dt \end{matrix} \right| = \int t dt = \frac{t^2}{2} = \frac{\sin^2 x}{2} + d_2$ So:

$c_2 = \frac{\sin^2 x}{2} + d_2$ Now replace:

$c_1 = -\frac{1}{2}x+\frac{1}{4}\sin 2x + d_1$ **and** $c_2 = \frac{\sin^2 x}{2} + d_2$ **in** $y_H = c_1 e^x \cos x + c_2 e^x \sin x$

y = $(-\frac{1}{2}x+\frac{1}{4}\sin 2x + d_1)\ e^x \cos x + (\frac{\sin^2 x}{2} + d_2)\ e^x \sin x$ **final solution**

7. Determine the particular solution for differential equations: $x^2y`` - xy` + y = 2x$

that satisfy the initial conditions y (1) = 0 and y `(1) = 1

Solution:

This is Euler's equation (see theoretical notes)

Introduce replacement : $x=e^t$, from here is: $y` = \frac{y`_t}{e^t}$; $y`` = \frac{y_t`` - y_t`}{e^{2t}}$;

$x^2y``-xy`+y = 2x$

$$e^{2t}\frac{y_t^{``} - y_t^{`}}{e^{2t}} - e^t \frac{y_t^{`}}{e^t} + y_t = 2e^t$$

$$y_t^{``} - y_t^{`} - y_t^{`} + y_t = 2e^t$$

$y_t^{``} - 2y_t^{`} + y_t = 2e^t$ this is nonhomogenous linear differential equations

$$y_t'' - 2y_t' + y_t = 0$$

$$p^2 - 2p + 1 = 0$$

$$p_{1,2} = \frac{2 \pm 0}{2} \Rightarrow p_1 = 1, p_2 = 1$$

$$y_t(H) = c_1 e^t + c_2 t e^t$$

method variation of parameters:

$c`_1 e^t + c`_2 t e^t = 0$
$c`_1 e^t + c`_2 (e^t + t e^t) = 2e^t$ both equations divide with e^t

$c`_1 + c`_2 t = 0$
$c`_1 + c`_2 (1+t) = 2$

$c`_1 = -c`_2 t$ We have expressed an unknown and change that to another equation…
$-c`_2 t + c`_2 (1+t) = 2$

$c`_2 = 2$
$c_2 = \int 2dt = 2t + d_2$

$c_2 = 2t + d_2$ We find one solution

$c`_1 = -c`_2 t =$ -2 t

$$c_1 = -2\int t dt = -2\frac{t^2}{2} + d_1 = -t^2 + d_1$$

$c_1 = -t^2 + d_1$ We find another solution

Go back…

$$y_t(H) = c_1 e^t + c_2 t e^t$$

$y_t = (-t^2 + d_1) e^t + (2t + d_2) t e^t$

$y_t = d_1 e^t + d_2 t e^t + t^2 e^t$ this is the final solution "by t", back in replacement **$x=e^t$** , or we can say: **t=lnx**

$y = d_1 x + d_2 x \ln x + x \ln^2 x$ **this is the final solution "by x"**

Next we will find required particular solution:

$y = d_1 x + d_2 x \ln x + x \ln^2 x$ here change **y(1)= 0**

$y` = d_1 + d_2(\ln x + 1) + \ln^2 x + 2\ln x$ here change **y`(1)= 1**

$0 = d_1 1 + d_2 1 \ln 1 + 1 \ln^2 1$ (ln1=0)

$1 = d_1 + d_2(\ln 1 + 1) + \ln^2 1 + 2\ln 1$

$0 = d_1 1 \Rightarrow d_1 = 0$

$1 = d_1 + d_2 \Rightarrow d_2 = 1$ return constants in the solution:

$y = 0x + 1x \ln x + x \ln^2 x$

$y = x \ln x + x \ln^2 x$ **this is demand particular solution!**

8. Solve the differential equation: **(x – 1)²y``- 2(x – 1)y` + 2y = (x – 1)²**

Solution:

This is also Euler's equation, replacement is **x – 1 = e^t ,** ⟶ **y`=**$\frac{y`_t}{e^t}$**; y``=**$\frac{y_t`` - y_t`}{e^{2t}}$

$(x-1)^2 y`` - 2(x-1)y` + 2y = (x-1)^2$

$e^{2t}\frac{y_t`` - y_t`}{e^{2t}} - 2e^t \frac{y_t`}{e^t} + 2y_t = e^{2t}$

$y_t`` - y_t` - 2y_t` + 2y_t = e^{2t}$

$y_t`` - 3y_t` + 2y_t = e^{2t}$

$y_t`` - 3y_t` + 2y_t = 0$

$p^2 - 3p + 2 = 0$

$p_{1,2} = \frac{3 \pm 1}{2} \Rightarrow p_1 = 2, p_2 = 1$

$y_t(H) = c_1 e^t + c_2 e^{2t}$ ⟶ $c_1 = c_1(t)$ and $c_2 = c_2(t)$

$c`_1 e^t + c`_2 e^{2t} = 0$ all divide with e^t

$c`_1 e^t + 2c`_2 e^{2t} = e^{2t}$

$c`_1 + c`_2 e^t = 0$ multiply this equation with (-1)
$c`_1 + 2c`_2 e^t = e^t$

$-c`_1 - c`_2 e^t = 0$
$c`_1 + 2c`_2 e^t = e^t$

$c_2^` e^t = e^t$

$c_2^` = 1$

$c_2 = \int 1dt$ So: $c_2 = t + d_2$ one solution

$c`_1 + c`_2 e^t = 0$

$c`_1 = -c`_2 e^t$ pa je $c`_1 = -e^t$ $\longrightarrow$ $c_1 = -e^t + d_1$

We find values : $c_2 = t + d_2$ and $c_1 = -e^t + d_1$, that we will return in homogeneous solution:

$y_t(H) = c_1 e^t + c_2 e^{2t}$

$y_t = (-e^t + d_1)e^t + (t + d_2)e^{2t}$

$y_t = -e^{2t} + d_1 e^t + te^t + d_2 e^{2t}$

$y_t = d_1 e^t + d_2 e^{2t} + te^{2t} - e^{2t}$ $\longrightarrow$ **$x - 1 = e^t$ or as we know $t = \ln(x-1)$**

$y = d_1(x-1) + d_2(x-1)^2 + (x-1)^2 \ln(x-1) - (x-1)^2$ **this is a general solution**

9. Solve the differential equation: $xy`` - (x+1)y` + y = 0$ if you know one particular solution $y_1 = e^x$

Solution:

$xy`` - (x+1)y` + y = 0$ divide all with x

$$y`` - \frac{x+1}{x} y` + \frac{1}{x} y = 0$$

To remind you a little theory:

Equation that has the form : y``+ a(x)y`+b(x)y=f(x)

Look at the appropriate homogeneous equation: **y``+ a(x)y`+b(x)y=0**

If you know one particular solution $y_1(x)$ of this equation, then another solution we can find:

$y_2(x)= y_1(x)\int \frac{e^{-\int a(x)dx}}{y_1^2(x)}dx$, and the solution of homogeneous equation will be: $y(x)=c_1\, y_1(x)+c_2\, y_2(x)$

Then solve home inhomogenous equation by **undetermined coefficients** or by **variation of parameters** .

We have a homogeneous equation, and you do not need to vary constant!

a(x)= $-\frac{x+1}{x}$ and $b(x) = \frac{1}{x}$

$$y_2(x)= y_1(x)\int \frac{e^{-\int a(x)dx}}{y_1^2(x)}dx$$

$$-\int a(x)dx = \int \frac{x+1}{x}dx = \int (\frac{x}{x}+\frac{1}{x})dx = x+\ln x$$

$$y_2(x)= e^x\int \frac{e^{x+\ln x}}{e^{2x}}dx = e^x\int \frac{e^x e^{\ln x}}{e^{2x}}dx = e^x\int \frac{x}{e^x}dx = \begin{vmatrix} x=u & e^{-x}dx=dv \\ dx=du & -e^{-x}=v \end{vmatrix} = e^x(-xe^{-x}-e^{-x}) = -x-1$$

$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$

$y(x) = c_1\, e^x + c_2\,(-x-1)$ is the final solution