Differential equations of the first-order (examples)

1. **Solve the differential equation: $x(1+y^2) = y\,y'$**

Solution:

$x(1+y^2) = y\,y'$

$x(1+y^2) = y\,\frac{dy}{dx}$ all multiply with dx ($dx \neq 0$) and divide with $1+y^2$

$x\,dx = \frac{ydy}{1+y^2}$ this is a differential equation that separates the variable!

$$\int xdx = \int \frac{ydy}{1+y^2}$$

$$\frac{x^2}{2} = \int \frac{ydy}{1+y^2} = \begin{vmatrix} 1+y^2 = t \\ 2ydy = dt \end{vmatrix} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + c = \frac{1}{2}\ln\left|1+y^2\right| + c$$

So:

$$\boxed{\frac{x^2}{2} = \frac{1}{2}\ln\left|1+y^2\right| + c}$$ the general solution of this differential equations (implicit form of solution)

2. **Solve the differential equation: $x^2 = 3\,y^2\,y'$**

Solution:

$x^2 = 3y^2 y'$

$x^2 = 3y^2 \frac{dy}{dx}$ all multiply with dx ($dx \neq 0$)

$x^2dx = 3y^2dy$ differential equation that separates the variable

$\int x^2 dx = \int 3y^2 dy$ both are table

$$\frac{x^3}{3} = 3\frac{y^3}{3} + c$$

$$\boxed{\frac{x^3}{3} = y^3 + c}$$ general solution

3. Solve the differential equation: $y` = \frac{2x+y}{2x}$

Solution:

$$y` = \frac{2x+y}{2x}$$

$$y` = \frac{x(2+\frac{y}{x})}{2x}$$

$$y` = \frac{2+\frac{y}{x}}{2}$$ this is a homogeneous differential equation

replacement: $\frac{y}{x} = z \Rightarrow y = zx \Rightarrow y` = z`x + z$

$$z`x + z = \frac{2+z}{2}$$

$$z`x = \frac{2+z}{2} - z$$

$$z`x = \frac{2+z-2z}{2}$$

$$z`x = \frac{2-z}{2}$$ this is differential equation that separates the variable $z` = \frac{dz}{dx}$

$$\frac{dz}{dx}x = \frac{2-z}{2}$$

$$\frac{dz}{2-z} = \frac{1}{2}\frac{dx}{x}$$

$$\int \frac{dz}{2-z} = \int \frac{1}{2}\frac{dx}{x}$$

$-\ln|2-z| = \frac{1}{2}\ln|x| + \ln c$ **trick** is that when all the solutions are by ln , to add lnc instead of c

$$\ln|2-z|^{-1} = \ln|x|^{\frac{1}{2}} + \ln c$$

$$\ln|2-z|^{-1} = \ln|x|^{\frac{1}{2}}c$$

$$|2-z|^{-1} = |x|^{\frac{1}{2}}c$$

$\frac{1}{2-z} = \sqrt{x}c$ return replacement $\frac{y}{x} = z$

$\frac{1}{2-\frac{y}{x}} = \sqrt{x}c$ This is implicit form of solution, **if required by your professor,**

express y, and we have explicit solution.

4. Solve the differential equation: $xy^2dy = (x^3 + y^3)dx$

Solution:

$$xy^2dy = (x^3 + y^3)dx$$

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$$

$$y` = \frac{x^3(1 + \frac{y^3}{x^3})}{xy^2}$$

$$y` = \frac{x^2(1 + \frac{y^3}{x^3})}{y^2}$$

$$y` = \frac{(1 + \frac{y^3}{x^3})}{\frac{y^2}{x^2}}$$

$$y` = \frac{1 + (\frac{y}{x})^3}{(\frac{y}{x})^2}$$ homogeneous differential equation

replacement : $\frac{y}{x} = z \Rightarrow y = zx \Rightarrow y` = z`x + z$

$$z`x + z = \frac{1 + z^3}{z^2}$$

$$z`x = \frac{1 + z^3}{z^2} - z$$

$$z`x = \frac{1 + z^3 - z^3}{z^2}$$

$$z`x = \frac{1}{z^2} \longrightarrow z` = \frac{dz}{dx}$$

$$\frac{dz}{dx}x = \frac{1}{z^2}$$

$$z^2dz = \frac{dx}{x}$$

$$\int z^2dz = \int \frac{dx}{x}$$

$$\frac{z^3}{3} = \ln|x| + c \longrightarrow \frac{y}{x} = z$$

So: $$\boxed{\frac{(\frac{y}{x})^3}{3} = \ln|x| + c}$$ is general solution

5. Solve the differential equation: xy\` - x^2 +2y = 0

Solution:

xy\` - x^2 +2y = 0

xy\` + 2y = x^2 all divide with x ($x \neq 0$)

$y' + \frac{2}{x} y = x$ This is linear differential equation $p(x)=\frac{2}{x}$ and $q(x)= x$

$y = e^{-\int p(x)dx}(c + \int q(x)e^{\int p(x)dx} dx)$ formula

Firs , solve: $\int p(x)dx$

$$\int p(x)dx = \int \frac{2}{x} dx = 2 \ln|x| = \ln|x|^2$$

$$\int q(x)e^{\int p(x)dx} dx = \int xe^{\ln x^2} dx = \int xx^2 dx = \int x^3 dx = \frac{x^4}{4}$$

$$y = e^{-\int p(x)dx}(c + \int q(x)e^{\int p(x)dx} dx) = e^{-\ln x^2}[c + \frac{x^4}{4}] = \frac{1}{x^2}[c + \frac{x^4}{4}]$$ So:

$$y = \frac{1}{x^2}[c + \frac{x^4}{4}]$$ **explicit** form of solution

6. Solve the differential equation: y\` -2xy = (x – x^3) e^{x^2}

Solution:

$y' - 2xy = (x - x^3)e^{x^2}$ linear differential equation ⟶ $p(x)= -2x$ and $q(x)= (x - x^3)e^{x^2}$

$$\int p(x)dx = ?$$

$$\int p(x)dx = \int (-2x)dx = -2\int xdx = -2\frac{x^2}{2} = -x^2$$

$$\int q(x)e^{\int p(x)dx} dx = \int (x - x^3)e^{x^2} e^{-x^2} dx = \int (x - x^3)dx = \frac{x^2}{2} - \frac{x^4}{4}$$

$$y = e^{-\int p(x)dx}(c + \int q(x)e^{\int p(x)dx} dx) = e^{x^2}[c + \frac{x^2}{2} - \frac{x^4}{4}]$$

$$y = e^{x^2}[c + \frac{x^2}{2} - \frac{x^4}{4}]$$

7. Solve the differential equation: $y` \cos^2 x = tg\, x - y$ and find that particular solution that meets conditions: x = 0 and y = 0

Solution:

First, we solve the differential equation and then we will find a constant value, for given conditions.

$$y` \cos^2 x = tg\, x - y$$

$$y` \cos^2 x + y = tg\, x \quad \text{all divide with } \cos^2 x$$

$$y` + \frac{1}{\cos^2 x} y = \frac{tgx}{\cos^2 x} \quad \text{linear differential equation} \quad p(x) = \frac{1}{\cos^2 x} \ldots\ldots\ldots\ldots\ldots\ldots q(x) = \frac{tgx}{\cos^2 x}$$

As usual, first we solve integral: $\int p(x)dx$

$$\int p(x)dx = \int \frac{1}{\cos^2 x}dx = tg\, x$$

$$\int q(x)e^{\int p(x)dx}dx = \int \frac{tgx}{\cos^2 x}e^{tgx}dx = \left| \begin{matrix} tgx = t \\ \frac{1}{\cos^2 x}dx = dt \end{matrix} \right| = \int te^t dt = partial \ldots integration =$$

$$\left| \begin{matrix} t = u & e^t dt = dv \\ dt = du & e^t = v \end{matrix} \right| = te^t - e^t = tgxe^{tgx} - e^{tgx}$$

$$y = e^{-\int p(x)dx}\left(c + \int q(x)e^{\int p(x)dx}dx\right) = e^{-tgx}[c + tgxe^{tgx} - e^{tgx}]$$

$$\boxed{y = e^{-tgx}c + tgx - 1}$$ **explicit** form of solution

Change here x = 0 and y = 0.

$$0 = e^{-tg0}c + tg0 - 1$$
$$0 = c - 1$$

c = 1 now ,back this in the general solution $y = e^{-tgx}1 + tgx - 1 = e^{-tgx} + tgx - 1$

This is particular solution.

8. Solve the differential equation: $xy`-2x^2\sqrt{y}=4y$

Solution:

$$xy`-2x^2\sqrt{y}=4y$$

$$xy`-4y=2x^2\sqrt{y}$$

$$xy`-4y=2x^2y^{\frac{1}{2}}$$

$y`-\frac{4}{x}y=2x\ y^{\frac{1}{2}}$ this is the Bernoulli differential equation , n = $\frac{1}{2}$, replacement:

$$y^{1-n}=u$$

$$y^{\frac{1}{2}}=u$$

$$\frac{1}{2}y^{\frac{-1}{2}}y`=u`$$

$$\frac{y`}{y^{\frac{1}{2}}}=2u`$$

$y`-\frac{4}{x}y=2x\ y^{\frac{1}{2}}$ all devide with y$^{\frac{1}{2}}$

Back in the equation:

$$\frac{y`}{y^{\frac{1}{2}}}-\frac{4}{x}\frac{y}{y^{\frac{1}{2}}}=2x$$

$2u`-\frac{4}{x}u=2x$ all devide fith 2

$u`-\frac{2}{x}u=x$ This is now linear differential equation (by "u")

u(x) = $e^{-\int p(x)dx}(c+\int q(x)e^{\int p(x)dx}dx)$

$$\int p(x)dx=\int(-\frac{2}{x})dx=-2\ln|x|=\ln|x|^{-2}=\ln\frac{1}{x^2}$$

$$\int q(x)e^{\int p(x)dx}dx=\int xe^{\ln x^{-2}}dx=\int x\frac{1}{x^2}dx=\int\frac{1}{x}dx=\ln|x|$$

u(x) = $e^{-\int p(x)dx}(c+\int q(x)e^{\int p(x)dx}dx)=e^{\ln x^2}[c+\ln|x|]$

u(x) = $x^2[c+\ln|x|]$ back in replacement : $\sqrt{y}=u$

$$\sqrt{y}=x^2[c+\ln|x|]$$

y = $x^4[c+\ln|x|]^2$

9. Determine what solution differential equations $(x^2+y^2+2x)dx+2ydy=0$ satisfy the initial condition y(0)=1

Solution:

First, we solve the differential equation and then find a constant value for the condition y(0)=1.

$(x^2+y^2+2x)dx+2ydy=0$ all divide with dx

$x^2+y^2+2x+2yy`=0$ all divide with 2y

$$\frac{x^2+2x}{2y}+\frac{1}{2}y+y`=0$$

$y`+\frac{1}{2}y=\frac{x^2+2x}{2}y^{-1}$ this is the Bernoulli differential equation ⟶ n = - 1

$$y^{1-n}=u$$

replacement : $y^2=u$

$$2yy`=u`$$

$y`+\frac{1}{2}y=\frac{x^2+2x}{2}y^{-1}$ all multiply with 2 y

$$2yy`+y^2=x^2+2x$$

$u`+u=x^2+2x$ linear differential equation

$$u(x)=e^{-\int p(x)dx}\left(c+\int q(x)e^{\int p(x)dx}dx\right)$$

$$\int p(x)dx=\int 1dx=x$$

$$\int q(x)e^{\int p(x)dx}dx=-\int (x^2+2x)e^x dx=\left|\begin{matrix} x^2+2x=u & e^x dx=dv \\ (2x+2)dx=du & e^x=v\end{matrix}\right|=$$

$$-e^x(x^2+2x)+\int e^x(2x+2)dx=\left|\begin{matrix} 2x+2=u & e^x dx=dv \\ 2dx=du & e^x=v\end{matrix}\right|=$$

$$-e^x(x^2+2x)+[e^x(2x+2)-\int 2e^x dx]$$

$$-e^x(x^2+2x)+e^x(2x+2)-2e^x=$$

$$e^x(-x^2\cancel{-2x}\cancel{+2x}\cancel{+2}\cancel{-2})=-x^2e^x$$

$$u(x)=e^{-\int p(x)dx}\left(c+\int q(x)e^{\int p(x)dx}dx\right)$$

$$u(x)=e^{-x}[c-x^2e^x]=\boxed{e^{-x}\cdot c-x^2}$$

return replacement

$y^2 = e^{-x} \cdot c - x^2$ Solution

Put here x = 0 and y = 1

1 = c , and

$y^2 = e^{-x} - x^2$

10. Solve the differential equation: $(2xy + 3y^2)dx + (x^2 + 6xy - 3y^2)dy = 0$

Solution:

Check whether this is a total differential equation:

$P(x,y) = 2xy + 3y^2$

$Q(x,y) = x^2 + 6xy - 3y^2$

$\frac{\partial P}{\partial y} = 2x + 6y$ i $\frac{\partial Q}{\partial x} = 2x + 6y$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, this is total differential.

Formula is: $C = \int P(x,y)dx + \int [Q - \frac{\partial}{\partial y}\int P(x,y)dx]dy$

$$\int P(x,y)dx = \int (2xy + 3y^2)dx = 2y\frac{x^2}{2} + 3y^2x = yx^2 + 3y^2x$$

$$\frac{\partial}{\partial y}(yx^2 + 3y^2x) = x^2 + 6xy$$

$$c = yx^2 + 3y^2x + \int [x^2 + 6xy - 3y^2 - x^2 - 6xy]dy$$

$$c = yx^2 + 3y^2x + \int [-3y^2]dy$$

$c = yx^2 + 3y^2x - y^3$ general solution

11. **Solve the differential equation:** $(3x+2y+y^2)dx+(x+4xy+5y^2)dy=0$ **knowing that its integration factor has a form** $\lambda=\lambda(x+y^2)$**. Find solution which passes through the point M (-2.1)**

Solution:

If μ (x,y)= μ (w(x,y)) (see the theoretical part) then:

$$\int\frac{d\mu}{\mu}=\int\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}{P\frac{\partial w}{\partial y}-Q\frac{\partial w}{\partial x}}dw$$ use this formula to find integrating factor.

$(3x+2y+y^2)dx+(x+4xy+5y^2)dy=0$ from here is:

P(x,y)= $3x+2y+y^2$ $\quad \frac{\partial P}{\partial y}=2+2y$ $\quad$ w=x+y^2 $\quad \frac{\partial w}{\partial x}=1$ $\quad \frac{\partial w}{\partial y}=2y$

Q(x,y)= $x+4xy+5y^2$ $\quad \frac{\partial Q}{\partial x}=1+4y$

$$\int\frac{d\mu}{\mu}=\int\frac{1+4y-2-2y}{(3x+2y+y^2)2y-(x+4xy+5y^2)}dw$$

$$\int\frac{d\mu}{\mu}=\int\frac{2y-1}{2xy-x+2y^3-y^2}dw$$

$$\int\frac{d\mu}{\mu}=\int\frac{2y-1}{x(2y-1)+y^2(2y-1)}dw$$

$$\int\frac{d\mu}{\mu}=\int\frac{2y-1}{(2y-1)(x+y^2)}dw$$

$$\int\frac{d\mu}{\mu}=\int\frac{1}{(x+y^2)}dw$$ w=x+y^2

ln μ =ln(x+y^2) +ln c ,then is $\quad$ ln μ =ln(x+y^2)c ⟶ for c=1 is μ = x+y^2

Therefore, required integrating factor is μ = x+y^2 .

Multiply the whole equation with μ = x+y^2

$$(x+y^2)(3x+2y+y^2)dx+(x+y^2)(x+4xy+5y^2)dy=0$$

$$(3x^2+2xy+xy^2+3xy^2+2y^3+y^4)dx+(x^2+4x^2y+5xy^2+xy^2+4xy^3+5y^4)dy=0$$

$$(3x^2+2xy+4xy^2+2y^3+y^4)dx+(x^2+4x^2y+6xy^2+4xy^3+5y^4)dy=0$$

$$\frac{\partial P}{\partial y}=2x+8xy+6y^2+4y^3 \qquad \frac{\partial Q}{\partial x}=2x+8xy+6y^2+4y^3$$

C= $\int P(x,y)dx+\int[Q-\frac{\partial}{\partial y}\int P(x,y)dx]dy$

$\int P(x,y)dx =\int (3x^2+2xy+4xy^2+2y^3+y^4)dx = 3\frac{x^3}{3}+2y\frac{x^2}{2}+4y^2\frac{x^2}{2}+2y^3x+y^4x$

$\frac{\partial}{\partial y}(3\frac{x^3}{3}+2y\frac{x^2}{2}+4y^2\frac{x^2}{2}+2y^3x+y^4x)$ =$x^2+4x^2y+6xy^2+4xy^3$

C= $x^3+x^2y+2y^2x^2+2y^3x+y^4x+\int[(x^2+4x^2y+6xy^2+4xy^3+5y^4)$-($x^2+4x^2y+6xy^2+4xy^3$)]dy

C=$x^3+x^2y+2y^2x^2+2y^3x+y^4x+\int 5y^4dy$

C=$x^3+x^2y+2y^2x^2+2y^3x+y^4x+y^5$ is general solution

Solution which passes through the point M(-2,1) is :

C= -8 + 4 + 4 – 4 – 2 + 1= - 5 ⟶ $x^3+x^2y+2y^2x^2+2y^3x+y^4x+y^5$ = - 5

12. Solve the differential equation: $y`=\frac{3y^3-2xy^2}{7-3xy^2}$ **if you know that the integration factor is in the function of y**

Solution:

$$y`=\frac{3y^3-2xy^2}{7-3xy^2}$$

$$\frac{dy}{dx}=\frac{3y^3-2xy^2}{7-3xy^2}$$

$$(7-3xy^2)dy=(3y^3-2xy^2)dx$$

$$(7-3xy^2)dy-(3y^3-2xy^2)dx=0$$

$$(2xy^2 - 3y^3)dx + (7 - 3xy^2)dy = 0$$

$$\frac{\partial P}{\partial y} = 4xy - 9y^2 \qquad \frac{\partial Q}{\partial x} = -3y^2$$

How is the integration factor in the function of y, we will use formula :

μ (x,y)= μ (y)

$$\int \frac{d\mu}{\mu} = \int \frac{1}{P}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dy$$

$$\int \frac{d\mu}{\mu} = \int \frac{1}{y^2(2x-3y)}(-3y^2 - 4xy + 9y^2)dy$$

$$\int \frac{d\mu}{\mu} = \int \frac{1}{y^2(2x-3y)}(-4xy + 6y^2)dy$$

$$\int \frac{d\mu}{\mu} = \int \frac{1}{y^2(2x-3y)} 2y(3y-2x)dy$$

$$\int \frac{d\mu}{\mu} = \int \frac{-2}{y}dy$$

$\ln|\mu| = -2\ln|y| + \ln c \quad \ln|\mu| = \ln|y|^{-2} + \ln c$ za c=1 je

$\mu = \frac{1}{y^2}$ the integration factor

$$\frac{1}{y^2}(2xy^2 - 3y^3)dx + \frac{1}{y^2}(7 - 3xy^2)dy = 0$$

$$(2x - 3y)dx + (\frac{7}{y^2} - 3x)dy = 0$$

$$\frac{\partial P}{\partial y} = -3 \qquad \frac{\partial Q}{\partial x} = -3$$

$\int P(x,y)dx = \int$ (2x – 3y)dx = x^2 – 3yx

$\frac{\partial}{\partial y}$(x^2 – 3yx) = - 3x

C= $\int P(x,y)dx + \int [Q - \frac{\partial}{\partial y}\int P(x,y)dx]dy$

C= x^2 - 3xy + $\int (\frac{7}{y^2} - 3x + 3x)dy$

C= x^2 - 3xy + $\int (\frac{7}{y^2})dy$

$C = x^2 - 3xy - \frac{7}{y}$ this is a general solution

13. Solve the differential equation **y` = ln(xy` - y)**

Solution:

Replacement: y`= p ⟶ $\frac{dy}{dx} = p \Rightarrow dy = pdx$

y` = ln(xy` - y)

p = ln(xp – y) here express y

$e^p = xp - y$
$y = px - e^p$ / d

$$dy = \frac{\partial(xp - e^p)}{\partial x}dx + \frac{\partial(xp - e^p)}{\partial p}dp$$

$$dy = pdx + (x - e^p)dp$$

$$pdx = pdx + (x - e^p)dp$$

$$(x - e^p)dp = 0$$

$$\int (x - e^p)dp = 0$$

$$xp - e^p + c = o$$

$$x = \frac{e^p - c}{p}$$

$$y = xp - e^p$$

$$y = \frac{e^p - c}{p}p - e^p$$

$$y = -c$$

$$\left.\begin{aligned} x &= \frac{e^p - c}{p} \\ y &= -c \end{aligned}\right\}$$ **general solution in the form of parametric**

14. Solve the differential equation $y`+y = xy`^2$

Solution:

And here we, as well as in the previous example, use a method with parameter:

y\`= p $\frac{dy}{dx} = p \Rightarrow dy = pdx$

$$y` + y = xy`^2$$
$$p + y = xp^2$$
$$y = xp^2 - p$$

$$dy = p^2 dx + (2px - 1)dp$$

$$pdx = p^2 dx + (2px - 1)dp$$

$(p - p^2)dx = (2px - 1)dp$ all divide with dp

$$(p - p^2)\frac{dx}{dp} = (2px - 1)$$

$$(p - p^2)x` = (2px - 1)$$

$(p - p^2)x` - 2px = -1$ multiply with -1

$p(p - 1)x` + 2px = 1$ all divide with p(p – 1)

$$x` + \frac{2p}{p(p-1)}x = \frac{1}{p(p-1)}$$

$x` + \frac{2}{(p-1)}x = \frac{1}{p(p-1)}$ this is linear differential equation \`\`by x\`\`, x=x(p)

Solve by using well-known formula:

x(p)= $e^{-\int p(p)dp}(c + \int q(p)e^{\int p(x)dp}dp)$

x(p) = $\frac{1}{(p-1)^2}[c + p - \ln|p|\,]$ This solution replace in y = xp^2- p

y(p)= $\frac{1}{(p-1)^2}[c + p - \ln|p|]p^2$- p

And this is a general solution in parametric form.

14. Show that the differential equation $(x^2+x)y`+y^2+(1-2x)y-2x=0$ has a particular solution $y_1=a$ where a is constant that should be set. Find the general solution.

Solution:

$(x^2+x)y`+y^2+(1-2x)y-2x=0$ has a particular solution $y_1=a \Rightarrow y_1`=0$ replace in equation:

$$0+a^2+(1-2x)a-2x=0$$
$$a^2+a-2ax-2x=0$$
$$x(-2a-2)+(a^2+a)=0$$

From here must be: -2a-2 = 0 and a^2+ a = 0

-2a = 2 a(a + 1)= 0

a= - 1 a=0 or a = - 1

Therefore, conclude that a = - 1 and one solution is $y_1=-1$

This is Riccati differential equations, form is y` = P(x) y^2 + Q(x)y + R(x)

If it is known one particular solution $y_1(x)$, then take the replacement: $y(x) = y_1(x)+\frac{1}{z(x)}$

$$y(x) = y_1(x)+\frac{1}{z(x)} \longrightarrow y=-1+\frac{1}{z} \Rightarrow y`=-\frac{z`}{z^2}$$

$$(x^2+x)y`+y^2+(1-2x)y-2x=0$$

$$(x^2+x)(-\frac{z`}{z^2})+(\frac{1}{z}-1)^2+(1-2x)(\frac{1}{z}-1)-2x=0$$

$$z`+\frac{2x+1}{x^2+x}z=\frac{1}{x^2+x}$$ this is linear differential equation “by z”

$$z(x) = e^{-\int p(x)dx}(c+\int q(x)e^{\int p(x)dx}dx)$$

$$\int p(x)dx=\int\frac{2x+1}{x^2+x}dx=\left|\begin{matrix}x^2+x=t\\(2x+1)dx=dt\end{matrix}\right|=\int\frac{dt}{t}=\ln|t|=\ln|x^2+x|$$

$$\int q(x)e^{\int p(x)dx}dx = \int \frac{1}{x^2+x}e^{\ln|x^2+x|}dx = \text{x}$$ pa je

$$z(x) = \frac{c+x}{x^2+x}$$ return replacement and

$$\frac{1}{y+1} = \frac{c+x}{x^2+x}$$

$$\boxed{y = \frac{x^2-c}{x+c}}$$ **general solution**

15. We have differential equation $xy` = y^2 - (2x+1)y + x^2 + 2x$

Determine the real numbers a and b so that y = ax + b is particular solution of the equation and then find the general solution.

Solution:

$y = ax + b \Rightarrow y` = a$ replace in given differential equation

$$xy` = y^2 - (2x+1)y + x^2 + 2x$$

$$xa = (ax+b)^2 - (2x+1)(ax+b) + x^2 + 2x$$

$$0 = a^2x^2 + 2abx + b^2 - 2ax^2 - 2bx - ax - b + x^2 + 2x - ax$$

$x^2(a^2 - 2a + 1) + x(2ab - 2b - 2a + 2) + b^2 - b = 0$ from here must be:

$a^2 - 2a + 1 = 0$ and $2ab - 2b - 2a + 2 = 0$ and $b^2 - b = 0$

$(a-1)^2 = 0$ $(2b-2)(a-1) = 0$ $b(b-1) = 0$

a = 1 a= 1 or b= 1 b= 0 or b= 1

In this way, we have received two particular solutions:: y = x and y = x+1

We will of course choose the easier : y = x for another part of the task.

$xy` = y^2 - (2x+1)y + x^2 + 2x$ This is Riccati differential equations, replacement is:

$y(x) = y_1(x) + \frac{1}{z(x)}$ then $y = x + \frac{1}{z} \Rightarrow y` = 1 - \frac{z`}{z^2}$ replace in $xy` = y^2 - (2x+1)y + x^2 + 2x$

$$x(1 - \frac{z`}{z^2}) = (x + \frac{1}{z})^2 - (2x+1)(x + \frac{1}{z}) + x^2 + 2x$$

$z`-\frac{1}{x}z=-\frac{1}{x}$ this is linear differential equation

z(x) = $e^{-\int p(x)dx}(c+\int q(x)e^{\int p(x)dx}dx)$

z(x) = xc+1 return replacement $y=x+\frac{1}{z}\Rightarrow\frac{1}{z}=y-x\Rightarrow z=\frac{1}{y-x}$

$$\frac{1}{y-x}=xc+1$$

$$y-x=\frac{1}{xc+1}$$

$$\boxed{y=x+\frac{1}{xc+1}}$$ **general solution**

16. Solve the differential equation: $x^2y`=x^2y^2+xy+1$

Solution:

$x^2y`=x^2y^2+xy+1$ all divide with x^2

$y`=y^2+\frac{1}{x}y+\frac{1}{x^2}$ This is Riccati d.e. y` = P(x) y^2 + Q(x)y + R(x)

Replacement z = yx where is z = z(x) (see theoretical note ...)

$$z=yx\Rightarrow z`=y`x+y\Rightarrow y`=\frac{z`-y}{x}$$

$$y`=y^2+\frac{1}{x}y+\frac{1}{x^2}$$

$\frac{z`-y}{x}`=(\frac{z}{x})^2+\frac{1}{x}\frac{z}{x}+\frac{1}{x^2}$ all multiply with x^2

$$x(z`-y)=z^2+z+1$$

$xz`-xy=z^2+z+1$ replace that yx = z

$$xz`-z=z^2+z+1$$

$xz` = z^2 + 2z + 1$ differential equation that separates the variable $z` = \frac{dz}{dx}$

$$x\frac{dz}{dx} = z^2 + 2z + 1 \longrightarrow \frac{dz}{(z+1)^2} = \frac{dx}{x}$$

$-\frac{1}{z+1} = \ln|x| + c$ return replacement z = xy :

$$\boxed{-\frac{1}{yx+1} = \ln|x| + c}$$ **general solution**