MULTIPLE INTEGRALS - TASKS (VI PART)

Triple integrals

If the function *f (x, y, z)* is continuous in *V,* which is determined by

$$V:\begin{cases} x_1 \le x \le x_2 \\ y_1(x) \le y \le y_2(x) \\ z_1(x,y) \le z \le z_2(x,y) \end{cases} \qquad \text{then} \qquad \iiint\limits_V f(x,y,z)dxdydz = \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

Some professors prefer a different record:

$$\iiint\limits_V f(x,y,z)dxdydz = \int_{x_1}^{x_2} \left(\int_{y_1(x)}^{y_2(x)} \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \right) dy \right) dx$$

You certainly do what your teacher requires.

In fact, both records indicate that you first solve the integral "by z", then"by y", and at the end “by x”
This rank in the integration is not required. Depending on the specific situation we can change order of integration…

$$\iiint\limits_V f(x,y,z)dxdydz = \int_{x_1}^{x_2} \left(\int_{z_1(x)}^{z_2(x)} \left(\int_{y_1(x,z)}^{y_2(x,z)} f(x,y,z)dy \right) dz \right) dx$$

This would mean that first we work “by y", then" by z" and at the end “by x". And so on ..

Example 1.

Calculate the value of the triple integral: $\int_0^1 dx \int_0^2 dy \int_0^3 dz$.

Solution:

Here we have given the limits and the order of integration, just to solve

So, first solve $\int_0^1 dx \int_0^2 dy \boxed{\int_0^3 dz}$

$$\int_0^1 dx \int_0^2 dy \int_0^3 dz = \int_0^1 dx \int_0^2 z\Big|_0^3 dy = \int_0^1 dx \int_0^2 3dy = 3\int_0^1 dx \underbrace{\boxed{\int_0^2 dy}}_{\text{Now this}} = 3\int_0^1 y\Big|_0^2 dx = 3\int_0^1 2dx = 6\int_0^1 dx = 6\cdot 1 = 6$$

Or, we can write:

$$\int_0^1 dx\int_0^2 dy\int_0^3 dz = \int_0^1 \left(\int_0^2\left(\int_0^3 dz\right)dy\right)dx = \int_0^1 \left(\int_0^2 3dy\right)dx = \int_0^1 6dx = 6$$

Example 2.

Calculate the value of the triple integral : $\iiint_V (x+y+z)dxdydz$ if V is given by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

<u>*Solution:*</u>

First, we do " by z"in this situation, x and y are viewed as constants.

$$\iiint_V (x+y+z)dxdydz =$$

$$\int_0^1 \left(\int_0^1\left(\int_0^1 (x+y+z)dz\right)dy\right)dx = \int_0^1 \left(\int_0^1\left(xz+yz+\frac{z^2}{2}\right)\Bigg|_0^1 dy\right)dx = \int_0^1 \left(\int_0^1\left(x+y+\frac{1}{2}\right)dy\right)dx$$

Now solve "by y" and x treated as a constant.

$$\int_0^1 \left(\int_0^1\left(x+y+\frac{1}{2}\right)dy\right)dx = \int_0^1 \left(xy+\frac{y^2}{2}+\frac{1}{2}y\right)\Bigg|_0^1 dx = \int_0^1 (x+\frac{1}{2}+\frac{1}{2})dx = \int_0^1 (x+1)dx$$

Finally we solve the ordinary integral "by x":

$$\int_0^1 (x+1)dx = \left(\frac{x^2}{2}+x\right)\Bigg|_0^1 = \frac{1}{2}+1 = \boxed{\frac{3}{2}}$$

Example 3.

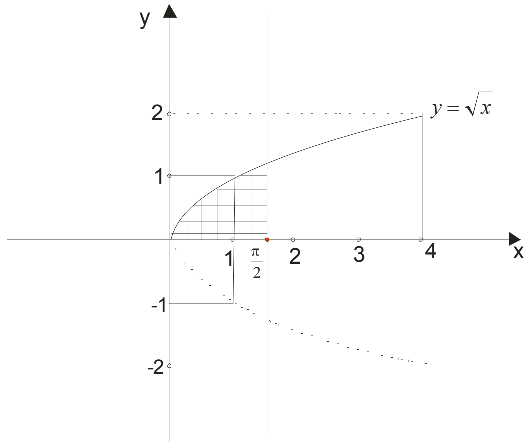
Calculate the value of the triple integral: $\iiint_V y\cos(z+x)dxdydz$ if V is limited with the cylinder $y=\sqrt{x}$ and with planes y = 0, z = 0 and $x+z=\frac{\pi}{2}$.

<u>*Solution:*</u>

Here , we do not have borders, so we must first to determine them:

$x+z=\frac{\pi}{2} \to z=\frac{\pi}{2}-x$ and we have $0 \le z \le \frac{\pi}{2}-x$

To determine how "behave" x and y, we draw image in plane z = 0



It is clear that $0 \le y \le \sqrt{x}$, and $0 \le x \le \frac{\pi}{2}$

Now, we can solve integral :

$$\iiint_V y\cos(z+x)dxdydz = \int_0^{\frac{\pi}{2}}\left(\int_0^{\sqrt{x}}\left(\int_0^{\frac{\pi}{2}-x} y\cos(z+x)dz\right)dy\right)dx$$

First we do:

$$\int_0^{\frac{\pi}{2}-x} y\cos(z+x)dz = y\int_0^{\frac{\pi}{2}-x}\cos(z+x)dz = y\sin(z+x)\Big|_0^{\frac{\pi}{2}-x} = y\left(\sin(\frac{\pi}{2}-x+x)-\sin(0+x)\right)$$

$$= y\left(\sin\frac{\pi}{2}-\sin(x)\right) = y(1-\sin x)$$

Return to:

$$\iiint_V y\cos(z+x)dxdydz = \int_0^{\frac{\pi}{2}}\left(\int_0^{\sqrt{x}}\left(\int_0^{\frac{\pi}{2}-x} y\cos(z+x)dz\right)dy\right)dx = \int_0^{\frac{\pi}{2}}\left(\int_0^{\sqrt{x}} y(1-\sin x)dy\right)dx$$

Now:

$$\int_0^{\sqrt{x}} y(1-\sin x)dy = (1-\sin x)\int_0^{\sqrt{x}} ydy = (1-\sin x)\frac{y^2}{2}\Big|_0^{\sqrt{x}} = (1-\sin x)\frac{x}{2}$$

$$\iiint_V y\cos(z+x)dxdydz = \int_0^{\frac{\pi}{2}}\left(\int_0^{\sqrt{x}}\left(\int_0^{\frac{\pi}{2}-x} y\cos(z+x)dz\right)dy\right)dx = \int_0^{\frac{\pi}{2}}\left(\int_0^{\sqrt{x}} y(1-\sin x)dy\right)dx = \int_0^{\frac{\pi}{2}}(1-\sin x)\frac{x}{2}dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}(x - x\sin x)dx$$

Finally:

$$\iiint_V y\cos(z+x)dxdydz = \frac{1}{2}\int_0^{\frac{\pi}{2}}(x - x\sin x)dx = \boxed{\frac{\pi^2}{16}+\frac{1}{2}}$$

Example 4.

Calculate the value of the triple integral: $\iiint_V \sqrt{x^2+y^2}\,dxdydz$ if V is given with $x^2+y^2=z^2$ and $z = 1$.

Solution:

Find the intersection of the cone and the plane that will give us the limits:

$$x^2+y^2=z^2 \wedge z=1$$
$$x^2+y^2=1$$

Here is convenient to use:

Cylindrical coordinates:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \rightarrow |J| = r$$

then $\iiint_V f(x,y,z)dxdydz = \iiint f(r\cos\varphi, r\sin\varphi, z)drd\varphi dz = \int_{\varphi_1}^{\varphi_2} d\varphi \int_0^{r} rdr \int_{z_1}^{z_2} fdz$

Do not be confused, some professors do not take z = z, but rather that:

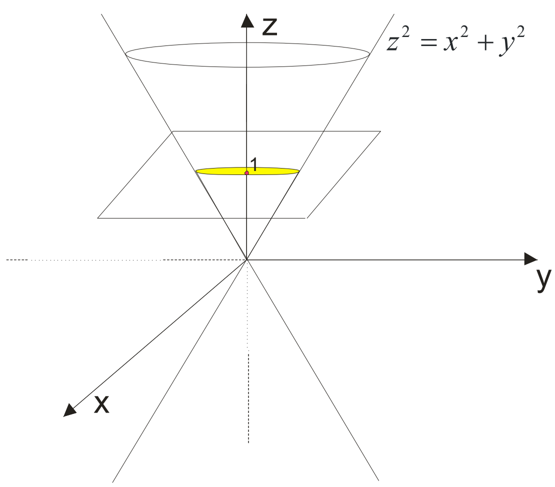
$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = h \end{cases} \rightarrow |J| = r$$, But it essentially does not change things ….

So:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \rightarrow |J| = r$$ and then $$\begin{aligned} & x^2 + y^2 = 1 \\ & (r\cos\varphi)^2 + (r\sin\varphi)^2 = 1 \\ & r^2(\cos^2\varphi + \sin^2\varphi) = 1 \\ & r^2 = 1 \rightarrow r = 1 \end{aligned}$$ and $0 \le r \le 1$.

Angle takes the values $0 \le \varphi \le 2\pi$.

Yet to determine the limits of z.



From above is plane and below is cone, so $r \le z \le 1$ because $\sqrt{x^2 + y^2} = \sqrt{r^2} = r$

$$\iiint_V \sqrt{x^2 + y^2}\,dxdydz = ?$$

$$\iiint_V \sqrt{x^2 + y^2}\,dxdydz = \int_0^{2\pi} d\varphi \int_0^1 dr \int_r^1 r \cdot r dz = \int_0^{2\pi} \left(\int_0^1 r^2 \left(\int_r^1 dz \right) dr \right) d\varphi = \int_0^{2\pi} \left(\int_0^1 r^2 (1 - r)\,dr \right) d\varphi =$$

$$= \int_0^{2\pi} \left(\int_0^1 \left(r^2 - r^3 \right) dr \right) d\varphi = 2\pi \left(\frac{r^3}{3} - \frac{r^4}{4} \right) \Bigg|_0^1 =$$

$$= \boxed{\frac{\pi}{6}}$$

Example 5.

Calculate the value of the triple integral: $\iiint\limits_V \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz$ if V is given with $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution:

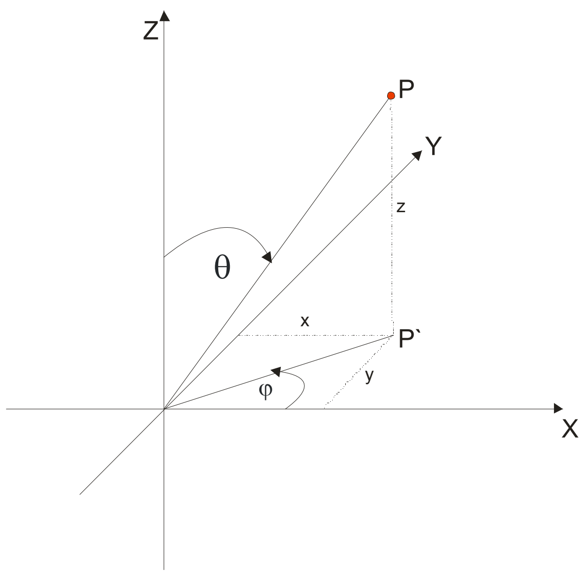
Here is convenient to use:
Spherical coordinates:

$$\begin{cases} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{cases}$$

$|J| = r^2\sin\theta$ From here we have $x^2+y^2+z^2 = r^2$

Angles φ and θ we determine from task and take care that the most common:

$r \geq 0; 0 \leq \varphi \leq 2\pi; 0 \leq \theta \leq \pi$



So : φ is angle in the plane z=0 and θ is angle in space …

We can use (depending on the situation) and modified spherical coordinates (generalized):

$$\begin{cases} x = ar\cos\varphi\sin\theta \\ y = br\sin\varphi\sin\theta \\ z = cr\cos\theta \end{cases}$$ and here is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=r^2$ and $|J| = abcr^2\sin\theta$

In our task, we use these little modified spherical coordinates, because it is an ellipsoid!

So:

$$\begin{cases} x = ar\cos\varphi\sin\theta \\ y = br\sin\varphi\sin\theta \\ z = cr\cos\theta \end{cases} \quad \text{and from here is} \quad |J| = abcr^2\sin\theta$$

Why is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=r^2$? To prove this:

$$\left.\begin{aligned} x &= ar\cos\varphi\sin\theta \\ y &= br\sin\varphi\sin\theta \\ z &= cr\cos\theta \end{aligned}\right\} \text{replace in } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

$$\frac{(ar\cos\varphi\sin\theta)^2}{a^2}+\frac{(br\sin\varphi\sin\theta)^2}{b^2}+\frac{(cr\cos\theta)^2}{c^2}=$$

$$\frac{\cancel{a^2}r^2\cos^2\varphi\sin^2\theta}{\cancel{a^2}}+\frac{\cancel{b^2}r^2\sin^2\varphi\sin^2\theta}{\cancel{b^2}}+\frac{\cancel{c^2}r^2\cos^2\theta}{\cancel{c^2}}=$$

$$r^2\cos^2\varphi\sin^2\theta+r^2\sin^2\varphi\sin^2\theta+r^2\cos^2\theta=$$

$$r^2\sin^2\theta\left(\cos^2\varphi+\sin^2\varphi\right)+r^2\cos^2\theta=$$

$$r^2\sin^2\theta+r^2\cos^2\theta=r^2\left(\sin^2\theta+\cos^2\theta\right)=r^2$$

We said that: $0\le r; 0\le\varphi\le 2\pi; 0\le\theta\le\pi$ so we have only correction: $0\le r\le 1;\quad 0\le\varphi\le 2\pi;\quad 0\le\theta\le\pi$

Because $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\to r^2=1\to r=1$

Now we can solve the integral:

$$\iiint_V\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz = abc\int_0^{\pi} d\theta\int_0^{2\pi} d\varphi\int_0^1 r^2\sin\theta r^2 dr = abc\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi} d\varphi\int_0^1 r^4 dr$$

Now this is not difficult tofind solution, we get $\iiint_V\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz = \boxed{\frac{4\pi}{5}abc}$

Calculate the volume using triple integrals

Volume calculated from the formula **V**= $\iiint\limits_V dxdydz$ by V

Example 1.

Find a volume of the body limited with surfaces: $z = x^2 + y^2$, $z = 2x^2 + 2y^2$, $y = x$ and $y = x^2$.

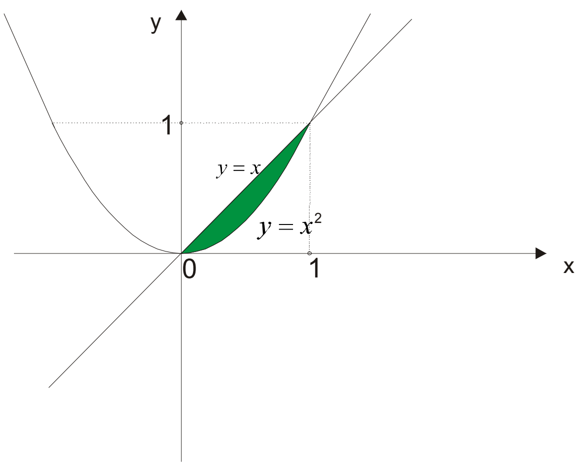
Solution:

This body is therefore limited by two paraboloids $z = x^2 + y^2$, $z = 2x^2 + 2y^2$, with plane $y = x$ and with cylinder $y = x^2$.

Formula is used $V = \iiint\limits_V dxdydz$

To determine the boundaries:

It is clear that $x^2 + y^2 \le z \le 2x^2 + 2y^2$, and for x and y we look at the picture :



$0 \le x \le 1$, $x^2 \le y \le x$

Now to find the volume:

$$V = \iiint\limits_V dxdydz = \int_0^1 dx \int_{x^2}^{x} dy \int_{x^2+y^2}^{2x^2+2y^2} dz = \int_0^1 dx \int_{x^2}^{x} z \Big|_{x^2+y^2}^{2x^2+2y^2} dy = \int_0^1 dx \int_{x^2}^{x} \left(x^2 + y^2\right) dy =$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)\Big|_{x^2}^{x} dx = \int_0^1 \left(\frac{4x^3}{3} - x^4 - \frac{x^6}{6}\right) dx$$

$$\boxed{V = \frac{3}{35}}$$

Example 2.

Find the volume of the body contained paraboloid $6 - z = x^2 + y^2$ and cone $z^2 = x^2 + y^2$

Solution:

Find the intersection:

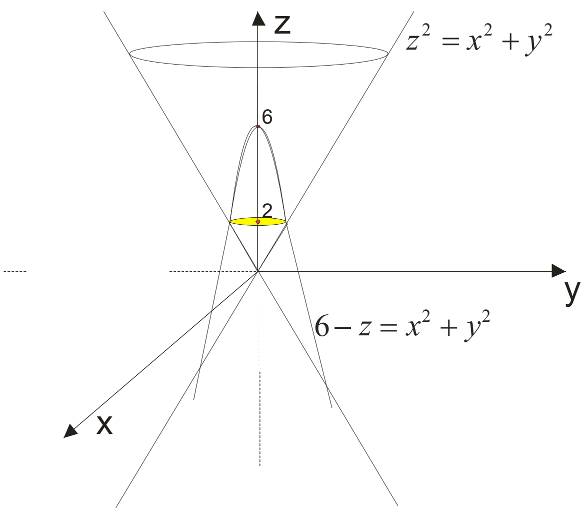
$z^2 = x^2 + y^2 \wedge 6 - z = x^2 + y^2$

$z^2 = 6 - z$

$z^2 + z - 6 = 0$

$\boxed{z_1 = 2}$

$z_2 = -3$



It is convenient to take the cylindrical coordinates:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \quad \rightarrow |J| = r \\ z = z \end{cases}$$

We have :

$x^2 + y^2 = 4 \rightarrow r^2 = 4 \rightarrow r = 2$

$\boxed{0 \le r \le 2}$

Angle : $\boxed{0 \le \varphi \le 2\pi}$

For paraboloid : $6 - z = x^2 + y^2 \rightarrow 6 - z = r^2 \rightarrow z = 6 - r^2$, for cone: $z^2 = x^2 + y^2 \rightarrow z^2 = r^2 \rightarrow z = r$,

So: $\boxed{r \le z \le 6 - r^2}$

Now we can calculate the volume:

$$V = \iiint_V dxdydz = \int_0^{2\pi} d\varphi \int_0^2 rdr \int_r^{4-r^2} dz$$

Here there are no problems, easy solved …

Solution: $\boxed{V = \frac{32\pi}{3}}$

Example 3.

Calculate the volume of the body that limits the surface $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1$

<u>*Solution:*</u>

This is one tricky situation when we have to use:

$$\begin{cases} x = ar\cos^{\beta}\varphi\sin^{\alpha}\theta \\ y = br\sin^{\beta}\varphi\sin^{\alpha}\theta \\ z = cr\cos^{\alpha}\theta \end{cases}$$

Jacobian in this situation is:

$$|J| = abcr^2 \cdot \alpha \cdot \beta \cdot \sin^{2\alpha-1}\theta\cos^{\alpha-1}\theta\sin^{\beta-1}\varphi\cos^{\beta-1}\varphi$$

The basic period is $r \geq 0; 0 \leq \varphi \leq 2\pi; 0 \leq \theta \leq \pi$

We will take:

$$x = ar\cos^4\varphi\sin^4\theta$$
$$y = br\sin^4\varphi\sin^4\theta$$
$$z = cr\cos^4\theta$$

Jacobian will be::

$$|J| = abcr^2 \cdot 4 \cdot 4 \cdot \sin^{2\cdot4-1}\theta\cos^{4-1}\theta\sin^{4-1}\varphi\cos^{4-1}\varphi$$
$$|J| = 16abcr^2\sin^7\theta\cos^3\theta\sin^3\varphi\cos^3\varphi$$

What are this replacement really good ?

$x = ar\cos^4\varphi\sin^4\theta; y = br\sin^4\varphi\sin^4\theta; z = cr\cos^4\theta$ replace in $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1$

$$\sqrt{\frac{ar\cos^4\varphi\sin^4\theta}{a}}+\sqrt{\frac{br\sin^4\varphi\sin^4\theta}{b}}+\sqrt{\frac{cr\cos^4\theta}{c}}=1$$

$$\sqrt{r}\cos^2\varphi\sin^2\theta+\sqrt{r}\sin^2\varphi\sin^2\theta+\sqrt{r}\cos^2\theta=1$$

$$\sqrt{r}\sin^2\theta\left(\cos^2\varphi+\sin^2\varphi\right)+\sqrt{r}\cos^2\theta=1$$

$$\sqrt{r}\sin^2\theta+\sqrt{r}\cos^2\theta=1$$

$$\sqrt{r}\left(\sin^2\theta+\cos^2\theta\right)=1$$

$$\sqrt{r}=1\rightarrow r=1\rightarrow \boxed{0\le r\le 1}$$

There is in fact the biggest problem to find the limits for the angles….

The basic period is here $r\ge 0; 0\le\varphi\le 2\pi; 0\le\theta\le\pi$, But it just tells us to what limits angles must be ...

In our task must be valid that:

$x\ge 0; y\ge 0; z\ge 0$, Means that the angles must be in the first quadrant, that is:

$0\le\varphi\le\frac{\pi}{2}$ and $0\le\theta\le\frac{\pi}{2}$

Now we can move on to solving the integral, that is, calculating the volume:

$$V=\iiint\limits_V dxdydz=\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\pi}{2}}d\theta\int_0^1 16abcr^2\sin^7\theta\cos^3\theta\sin^3\varphi\cos^3\varphi dr=$$

$$V=\iiint\limits_V dxdydz=\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\pi}{2}}d\theta\int_0^1 16abcr^2\sin^7\theta\cos^3\theta\sin^3\varphi\cos^3\varphi dr=$$

$$=16abc\int_0^{\frac{\pi}{2}}\sin^3\varphi\cos^3\varphi d\varphi\int_0^{\frac{\pi}{2}}\sin^7\theta\cos^3\theta d\theta\int_0^1 r^2 dr=$$

So, here we need to resolve three separate integrals, because the limits are not included in the second integral ...

$$\int\sin^3\varphi\cos^3\varphi d\varphi=?$$

Must make use knowledge of trigonometry ….

$$\cos^3\varphi=\cos^2\varphi\cdot\cos\varphi=(1-\sin^2\varphi)\cdot\cos\varphi=\cos\varphi-\sin^2\varphi\cos\varphi$$

$$\int \sin^3 \varphi \cos^3 \varphi d\varphi = \int \sin^3 \varphi \left(\cos\varphi - \sin^2 \varphi \cos\varphi \right) d\varphi =$$

$$\int (\sin^3 \varphi \cos\varphi - \sin^5 \varphi \cos\varphi) d\varphi =$$

etc.

Final solution will be: $\boxed{V = \frac{abc}{90}}$