Logarithms – definition and properties

$$\log_a b = x \Leftrightarrow b = a^x \quad , \quad (a > 0, a \neq 0)$$

Important: $b > 0$ is the most common condition, which we set. Also $a \in R, a \neq 1, and \quad a > 0$

b-is numerus (logarithmand), and a is base

The basic properties of logarithm:

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a (xy) = \log_a x + \log_a y$
4. $\log_a \frac{x}{y} = \log_a x - \log_a y$
5. $\log_a x^n = n \log_a x$
6. $\log_{a^s} x = \frac{1}{s} \log_a x$
7. $\log_a b \cdot \log_a a = 1 \rightarrow \qquad \log_a b = \frac{1}{\log_b a}$
8. To switch to a new base c: $\log_a b = \frac{\log_c b}{\log_c a}$
9. $a^{\log_a b} = b$

→ If the base is a = 10, such logarithms are called **decimal** : $\log_{10} x = \log 10$
(So, when there is no basis ,it means that is 10)

→ If the basis is a = e ($e \approx 2{,}7$),then such logarithms called **natural :** $\log_e x = \ln x$

Must be careful to record:

$(\log_a x)^2 = \log_a^2 x = \log_a x \cdot \log_a x$

$\log_a x^2 = \log_a x \cdot x = 2 \log_a x$

Discover the properties of logarithm through the following examples:

Calculate:

1)

$\log_5 1 = ?$

$\log_6 1 = ?$

$\log_{\frac{1}{2}} 1 = ?$

$\log 1 = ?$

$\ln 1 = ?$

$\log_5 1 = 0$

$\log_6 1 = 0$

$\log_{\frac{1}{2}} 1 = 0$

$\log 1 = 0$

$\ln 1 = 0$

All these logarithms have a solution 0 , because: $\log_a 1 = 0$

2)

$\log_{12} 12 = ?$

$\log_{\frac{2}{3}} \frac{2}{3} = ?$

$\log 10 = ?$

$\log e = ?$

$\log_{12} 12 = 1$

$\log_{\frac{2}{3}} \frac{2}{3} = 1$

$\log 10 = 1$

$\log e = 1$

All these logarithms for the solution have 1, because : $\log_a a = 1$

3)

a) $\log_6 2 + \log_6 3 = ?$

b) $\log_{30} 2 + \log_{30} 5 + \log_{30} 3 = ?$

Take property **3:** $\log_a (xy) = \log_a x + \log_a y$ So:

a) $\log_6 2 + \log_6 3 = \log_6 (2 \cdot 3) = \log_6 6 =$ (property 2.)=1

b) $\log_{30} 2 + \log_{30} 5 + \log_{30} 3 = \log_{30} (2 \cdot 5 \cdot 3) = \log_{30} 30 = 1$

4)

a) $\log_5 10 + \log_5 2 = ?$

b) $\log_2 20 + \log_2 10 = ?$

Take property: $\log_a x - \log_a y = \log_a \frac{x}{y}$

a) $\log_5 10 - \log_5 2 = \log_5 \frac{10}{2} = \log_5 5 = 1$

b) $\log_2 20 - \log_2 10 = \log_2 \frac{20}{10} = \log_2 2 = 1$

5)

a) $\log_2 8 = ?$

b) $\log_5 \frac{1}{125} = ?$ Here we use $\log_a x - \log_a y = \log_a \frac{x}{y}$

c) $\log_a \sqrt[5]{a^2} = ?$ Reminder: $\sqrt[m]{a^n} = \frac{n}{m}$ and $\frac{1}{a^n} = a^{-n}$

a)

$$\log_2 8 = \log_2 2^3 = 3\log_2 2 = 3 \cdot 1 = 3$$

b)

$$\log_5 \frac{1}{125} = \log_5 \frac{1}{5^3} = \log_5 5^{-3} = -3\log_5 = -3 \cdot 1 = -3$$

c)

$$\log_a \sqrt[5]{a^2} = \log_a a^{\frac{2}{5}} = \frac{2}{5}\log_a a = \frac{2}{5} \cdot 1 = \frac{2}{5}$$

6)

a) $\log_{81} 3 = ?$

b) $\log_{\sqrt{2}} 2 = ?$

c) $\log_{\sqrt{3}} 27 = ?$

a)

$$\log_{81} 3 = \log_{3^4} 3 = \frac{1}{4}\log_3 3 = \frac{1}{4} \cdot 1 = \frac{1}{4}$$

b)

$$\log_{\sqrt{2}} 2 = \log_{2^{\frac{1}{2}}} = \frac{1}{\frac{1}{2}}\log_2 2 = 2 \cdot 1 = 2$$

c)

$$\log_{\sqrt{3}} 27 = \log_{3^{\frac{1}{2}}} 3^3 = 3 \cdot \frac{1}{\frac{1}{2}}\log_3 3 = 3 \cdot 2 \cdot 1 = 6$$

7)

a) $\log_5 2 \cdot \log_2 5 = ?$

b) $\log 15 \cdot \log_{15} 10 = ?$

Solutions are $\log_5 2 \cdot \log_2 5 = 1$ and $\log 15 \cdot \log_{15} 10 = 1$, because $\log_a b \cdot \log_b a = 1$

8)

a) $\log_3 2 \cdot \log_4 3 \cdot \log_5 4 \cdot \log_6 5 \cdot \log_7 6 \cdot \log_8 7 = ?$

b) If $\log_5 2 = a$ and $\log_5 3 = b$, calculate $\log_{45} 100 = ?$

Here we apply the switch to the new basis: $\log_a b = \dfrac{\log_c b}{\log_c a}$

a)

Go, for example, to take the new basis10, then: $\log_3 2 = \dfrac{\log 2}{\log 3}$; $\log_4 3 = \dfrac{\log 3}{\log 4}$, …

So: $\log_3 2 \cdot \log_4 3 \cdot \log_5 4 \cdot \log_6 5 \cdot \log_7 6 \cdot \log_8 7 = \dfrac{\log 2}{\cancel{\log 3}} \cdot \dfrac{\cancel{\log 3}}{\cancel{\log 4}} \cdot \dfrac{\cancel{\log 4}}{\cancel{\log 5}} \cdot \dfrac{\cancel{\log 5}}{\cancel{\log 6}} \cdot \dfrac{\cancel{\log 6}}{\cancel{\log 7}} \cdot \dfrac{\cancel{\log 7}}{\log 8} =$

$= \dfrac{\log 2}{\log 8} = \log_8 2 = \log_{2^3} 2 = \dfrac{1}{3}\log_2 2 = \dfrac{1}{3} \cdot 1 = \dfrac{1}{3}$

b)

$\log_5 2 = a \quad \wedge \quad \log_5 3 = b \qquad \log_{45} 100 = ?$

$\log_{45} 100 =$ (here it is clear that the new basis must be 5) $= \dfrac{\log_5 100}{\log_5 45} =$

$= \dfrac{\log_5 10^2}{\log_5 (5 \cdot 9)} = \dfrac{2\log_5 10}{\log_5 5 \cdot \log_5 9} = \dfrac{2\log_5 (5 \cdot 2)}{1 + \log_5 3^2} = \dfrac{2(\log_5 5 + \log_5 2)}{1 + 2\log_5 3} =$

$= \dfrac{2(1 + \log_5 2)}{1 + 2\log_5 3} = \dfrac{2(1 + a)}{1 + 2b}$

9)

a) $3^{\log_3 81} = ?$
b) $10^{\log 5} = ?$

So: $3^{\log_3 81} = 81$ and $10^{\log 5} = 5$, because: $a^{\log_a b} = b$

Now , we know the basic properties , we will introduce other types of tasks:

1) Find logarithms for basis 10:

a) $A = \frac{x \cdot y}{z}$
b) $B = \frac{x^2 \cdot y^3}{z^5}$
c) $C = \frac{\sqrt[3]{x}}{\sqrt[5]{y^2} \cdot \sqrt{y}}$
d) $D = \sqrt[3]{5x^4 y^3}$

Solution:

a)

$$A = \frac{x \cdot y}{z}$$

$$\log A = \log \frac{xy}{z} = \log(xy) - \log z = \log x + \log y - \log z$$

b)

$$B = \frac{x^2 \cdot y^3}{z^5}$$

$$\log B = \log \frac{x^2 \cdot y^3}{z^5} = \log(x^2 \cdot y^3) - \log z^5 = \log x^2 + \log y^3 - \log z^5 =$$
$$= 2\log x + 3\log y - 5\log z$$

c)

$$C=\frac{\sqrt[3]{x}}{\sqrt[5]{y^2}\cdot\sqrt{y}}$$ **Take heed:** $\sqrt[m]{a^n}=a^{\frac{n}{m}}$, $\sqrt{a}=a^{\frac{1}{2}}$

$$\log C=\log\frac{\sqrt[3]{x}}{\sqrt[5]{y^2}\cdot\sqrt{y}}=\log\sqrt[3]{x}-\log\left(\sqrt[5]{y^2}\cdot\sqrt{y}\right)=\log x^{\frac{1}{3}}-\left(\log y^{\frac{2}{5}}+\log z^{\frac{1}{2}}\right)=$$

$$=\frac{1}{3}\log x-\frac{2}{5}\log y-\frac{1}{2}\log z$$

d)

$$D=\sqrt[3]{5x^4y^3}$$

$$D=\sqrt[3]{5x^4y^3}=\sqrt[3]{5}\sqrt[3]{x^4}\sqrt[3]{y^3}=5^{\frac{1}{3}}\cdot x^{\frac{4}{3}}\cdot y$$

$$\log D=\log\left(5^{\frac{1}{3}}\cdot x^{\frac{4}{3}}\cdot y\right)$$

$$=\frac{1}{3}\log 5+\frac{4}{3}\log x+\log y$$

2) Solve the equations ("by x"):

a) $\log x=\log 4+2\log 5+\log 6-\log 15$

b) $\log x+\log 3=2\log r+\log\pi+\log H$

c) $2\log x-3\log a=\log 5+\log b+\frac{1}{2}\log c$

Solution:

a)

$$\log x=\log 4+2\log 5+\log 6-\log 15$$

$$\log x=\log\frac{4\cdot 25\cdot 6}{15}$$

$$\log x=\log\frac{600}{15}$$

$$\log x=\log 40$$

$$x=40$$

b)

$$\log x + \log 3 = 2\log r + \log \pi + \log H$$
$$\log(x \cdot 3) = \log r^2 + \log \pi + \log H$$
$$\log(3x) = \log r^2 \pi H$$
$$3x = r^2 \pi H$$
$$x = \frac{r^2 \pi H}{3}$$

c)

$$2\log x - 3\log a = \log 5 + \log b + \frac{1}{2}\log c$$
$$\log x^2 - 3\log a^3 = \log 5 + \log b + \log c^{\frac{1}{2}}$$
$$\log \frac{x^2}{a^3} = \log 5 \cdot b \cdot \sqrt{c}$$
$$\frac{x^2}{a^3} = 5b\sqrt{c}$$
$$x^2 = 5a^3 b\sqrt{c}$$
$$x = \sqrt{5a^3 b\sqrt{c}}$$

3) If $\log_{14} 7 = a$ **and** $\log_{14} 5 = b$ **Calculate** $\log_{35} 28 = ?$

Solution:

This is the type of tasks where we need to take a new basis, of course, it will be 14.

$$\log_{35} 28 = \frac{\log_{14} 28}{\log_{14} 35} = \frac{\log_{14} \frac{196}{7}}{\log_{14}(7 \cdot 5)} = \frac{\log_{14} 196 - \log_{14} 7}{\log_{14} 7 + \log_{14} 5} = \frac{\log_{14} 14^2 - \log_{14} 7}{\log_{14} 7 + \log_{14} 5} =$$
$$= \frac{2\log_{14} 14 - \log_{14} 7}{\log_{14} 7 + \log_{14} 5} = \frac{2-a}{a+b}$$