Systems square equation with two unknown

There are several types of systems:

1) The system of one square and one linear equations with two unknown

From linear equation express x or y (whichever is easier), this change in quadratic equation and solve it…

Example 1. Solve the system:

$$\begin{aligned} &2x^2+2y^2+3x-2=0 \\ &x-2y=-2 \end{aligned}$$

Solution:

$$\begin{aligned} &2x^2+2y^2+3x-2=0 \\ &x-2y=-2 \end{aligned}$$

$\longrightarrow$ from here express x (easier) and change into the square equation

$$\begin{aligned}
&x=2y-2 \\
&2(2y-2)^2+2y^2+6y-6-2=0 \\
&2(4y^2-8y+4)+2y+6y-6-2=0 \\
&8y^2-16y+8+2y^2+6y-8=0 \\
&10y^2-10y=0/:10 \\
&y^2-y=0 \\
&y(y-1)=0 \\
&y_1=0 \quad \vee \quad y-1=0 \\
&\qquad\qquad\quad y_2=1 \qquad \Rightarrow \qquad x_1=2\cdot 0-2=-2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad x_2=2\cdot 1-2=0
\end{aligned}$$

Solutios are: $(x_1,y_1)=(-2,0)$ and $(x_2,y_2)=(0,1)$

Example 2. Solve the system:

$$\begin{aligned} &3x^2+2xy+2y^2+3x-4y=0 \\ &2x-y+5=0 \end{aligned}$$

Solution:

$$3x^2 + 2xy + 2y^2 + 3x - 4y = 0$$
$$2x - y + 5 = 0$$

$$y = 2x + 5$$
$$3x^2 + 2x(2x+5) + 2(2x+5)^2 + 3x - 4(2x+5) = 0$$
$$3x^2 + 4x^2 + 10x + 2(4x^2 + 20x + 25) + 3x - 8x - 20 = 0$$
$$7x^2 + 10x + 8x^2 + 40x + 50 + 3x - 8x - 20 = 0$$
$$15x^2 + 45x + 30 = 0 / :15$$
$$x^2 + 3x + 2 = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm 1}{2}$$

$a = 1$, $b = 3$, $c = 2$

$$x_1 = -1$$
$$x_2 = -2$$

Replace x_1 and x_2 in $y = 2x + 5$:

$$y_1 = 2(-1) + 5 = -2 + 5 = 3$$
$$y_2 = 2(-2) + 5 = -4 + 5 = 1$$

Solutions are: $(-1,3)$, $(-2,1)$

2) The system of two square equations, which contain only ax^2 and ay^2 and free members

This is system:
$$a_1x^2 + b_1y^2 = c_1$$
$$a_2x^2 + b_2y^2 = c_2$$

The easiest method to solve this system is "the opposite coefficients".

Example 1. Solve the system:
$$5x^2 - 6y^2 = 11$$
$$7x^2 + 3y^2 = 714$$

Solution:

$5x^2 - 6y^2 = 11$

$7x^2 + 3y^2 = 714 \rightarrow$ multiply with 2

$$\left.\begin{aligned} 5x^2 - 6y^2 &= 111 \\ 14x^2 + 6y^2 &= 1428 \end{aligned}\right\} +$$

$19x^2 = 1539$

$x^2 = 81$

$x = \pm\sqrt{81}$ x= ± 9 ⟶ $x_1 = +9$ i $x_2 = -9$

$7x^2 + 3y^2 = 714$

$7 \cdot 81 + 3y^2 = 714$

$567 + 3y^2 = 714$

$3y^2 = 714 - 567$

$3y^2 = 147$

$y^2 = 49$

$y = \pm\sqrt{49}$

$y_1 = +7$

$y_2 = -7$

Now, make"combination" : (9,7), (9,-7), (-9,7), (-9,-7) are solutions.

Before becoming acquainted with the new type of system, we have to learn what are **homogeneous** equations.

Its general form is: $Ax^2 + Bxy + Cy^2 = 0$

It can be solved easily with replacement: $x = yz \longrightarrow z = \frac{x}{y}$

$Ax^2 + Bxy + Cy^2 = 0$

$y^2\left(A\frac{x^2}{y^2} + B\frac{xy}{y^2} + C\frac{y^2}{y^2}\right) = 0$

$y^2\left(A\left(\frac{x}{y}\right)^2 + B\left(\frac{x}{y}\right) + C\right) = 0$

$y^2\left(Az^2 + Bz + C\right) = 0$

$y = 0 \quad \vee \quad Az^2 + Bz + C = 0 \longrightarrow$

$z_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$z_1 = \ldots$

$z_2 = \ldots$

$x = z_1 y$ i $x = z_2 y$

3) System of two square equations, which is one homogeneous

This system has form: $\begin{cases} Ax^2 + Bxy + Cy^2 = 0 \\ ax^2 + bxy + cy^2 + dx + ry + f = 0 \end{cases}$

Example 1. Solve the system: $x^2 - 3xy + 2y^2 = 0$

$$x^2 - 3x - y + 3 = 0$$

Solution:

$x^2 - 3xy + 2y^2 = 0 \rightarrow$ homogeneous, first solve it

$x^2 - 3x - y + 3 = 0$

$$x^2 - 3xy + 2y^2 = 0$$

$$y^2\underbrace{\left(\frac{x^2}{y^2} - 3\frac{x}{y} + 2\right)} = 0 \qquad \frac{x}{y} = z \quad \text{replacement} \quad x = zy$$

$$z^2 - 3z + 2$$

$$z_{1,2} = \frac{3 \pm 1}{2}$$

$$z_1 = 2$$

$$z_2 = 1$$

For $z_1 = 2 \Rightarrow x = 2y$

$$x^2 - 3x - y + 3 = 0$$

$$(2y)^2 - 3 \cdot 2y - y + 3 = 0$$

$$4y^2 - 6y - y + 3 = 0$$

$$4y^2 - 7y + 3 = 0$$

$$y_{1,2} = \frac{7 \pm 1}{8}$$

$$y_1 = 1$$

$$y_2 = \frac{6}{8} = \frac{3}{4}$$

$$x_1 = 2 \cdot 1 = 2, \; x_2 = 2 \cdot \frac{3}{4} = \frac{3}{2}$$

$(2,1)$, $\left(\frac{3}{2}, \frac{3}{4}\right)$

For $z_1 = 1 \Rightarrow x = y$

$$x^2 - 3x - y + 3 = 0$$

$$x^2 - 3x - x + 3 = 0$$

$$4y^2 - 4x + 3 = 0$$

$$x_{1,2} = \frac{4 \pm 2}{2}$$

$$x_1 = 3$$

$$x_2 = 1$$

$$y_1 = 3, \; y_2 = 1$$

$(3,3)$, $(1,1)$

Solutions are: (2,1), $\left(\frac{3}{2}, \frac{3}{4}\right)$**,(3,3), (1,1)**

Example 2. Solve the system:
$$\begin{aligned} x^2 - xy - 6y^2 &= 0 \\ x^2 - 2xy + 2y^2 &= 18 \end{aligned}$$

Solution:

and

$z^2 + z - 6 = 0$ replacement $\frac{x}{y} = z$

$$z_{1,2} = \frac{-1 \pm 5}{2}$$

$z_1 = 2$

$z_2 = -3$

$x = yz \Rightarrow \quad x = 2y \quad$ or $\quad x = -3y$

$$\begin{aligned} x &= 2y \\ x^2 - 2xy + 2y^2 &= 18 \end{aligned}$$

$(2y)^2 - 2 \cdot 2y \cdot y + 2y^2 = 18$

$4y^2 - 4y^2 + 2y^2 = 18$

$y^2 = 9$

$y = \pm 3$

$y_1 = +3 \qquad y_2 = -3$

$x_1 = 2 \cdot 3 \qquad x_2 = 2 \cdot (-3) = -6$

(6,3) (-6,-3)

$$\begin{aligned} x &= -3y \\ x^2 - 2xy + 2y^2 &= 18 \end{aligned}$$

$(-3y)^2 - 2 \cdot (-3y) \cdot y + 2y^2 = 18$

$9y^2 + 6y^2 + 2y^2 = 18$

$17y^2 = 18 \rightarrow$

$y^2 = \frac{18}{17}$

$y = \pm\sqrt{\frac{18}{17}}$

$y_1 = +\sqrt{\frac{18}{17}}$

$y_2 = -\sqrt{\frac{18}{17}}$

$x_1 = -3\sqrt{\frac{18}{17}} \qquad x_2 = 3\sqrt{\frac{18}{17}}$

$\left(-3\sqrt{\frac{18}{17}}, \sqrt{\frac{18}{17}}\right)$ and $\left(3\sqrt{\frac{18}{17}}, -\sqrt{\frac{18}{17}}\right)$

So, again there are four solutions!

4) Systems that reduce to the homogeneous equations

General form of this system is:

$$\begin{aligned} a_1x^2 + b_2xy + c_1y^2 &= d_1 \\ a_2x^2 + b_2xy + c_2y^2 &= d_2 \end{aligned}$$

The idea is "to destroyed" d_1 and d_2 by opposing coefficients, and to obtain homogeneous equation. Her resolve and create two new systems. **Nothing without examples**:

Example 1. Solve the system: $\begin{aligned} 2x^2 - 3xy + 2y^2 &= 4 \\ x^2 + xy + y^2 &= 7 \end{aligned}$

Solution:

$$\begin{aligned} 2x^2 - 3xy + 2y^2 &= 4 \\ x^2 + xy + y^2 &= 7 \end{aligned}$$

Multiply the first equation with 7, and the other with -4

$$\left.\begin{aligned} 14x^2 - 21xy + 14y^2 &= 28 \\ -4x^2 - 4xy - 4y^2 &= -28 \end{aligned}\right\} +$$

$$10x^2 - 25xy + 10y^2 = 0 / :5$$

$2x^2 - 5xy + 2y^2 = 0 \rightarrow$ We have received a homogeneous equation!

$2\frac{x^2}{y^2} - 5\frac{x}{y} + 2 = 0$ replacement $\frac{x}{y} = z$

$$2z^2 - 5z + 2 = 0$$

$$z_{1,2} = \frac{5 \pm 3}{4}$$

$$z_1 = 2$$

$$z_2 = \frac{1}{2}$$

$\frac{x}{y} = 2$ or $\frac{x}{y} = \frac{1}{2}$

$x = 2y$ or $y = 2x$

Now select one of the leading two equations (one with less numbers) and create two new system:

$$x = 2y$$
$$x^2 + xy + y^2 = 7$$

$$(2y)^2 + 2y \cdot y + y^2 = 7$$
$$4y^2 + 2y^2 + y^2 = 7$$
$$7y^2 = 7$$
$$y^2 = 1$$
$$y = \pm 1$$
$$y_1 = 1$$
$$y_2 = -1$$

$$x_1 = 2 \cdot y_1 = 2$$
$$x_2 = 2 \cdot (-1) = -2$$

$$y = 2x$$
$$x^2 + xy + y^2 = 7$$

$$x^2 + x \cdot 2x + (2x)^2 = 7$$
$$x^2 + 2x^2 + 4x^2 = 7$$
$$7x^2 = 7$$
$$x^2 = 1$$
$$x = \pm 1$$
$$x_1 = 1$$
$$x_2 = -1$$

$$x_1 = 2x_1 = 2$$
$$x_2 = 2 \cdot (-1) = -2$$

Finally solutions are: (2,1), (-2,-1), (1,2), (-1,-2)