Geometric progression(geometric sequence)

Head from the following two examples:

Example 1) 3, 6, 12, 24, 48 ...

Example 2) 81, 27, 9, 3, ...

If we carefully observe the **first example**, we notice that each next member of the progression we get when the previous multiply with 2. So,the next few members are $48*2=96,.....96*2=192,...$

In **example 2** note that each member ,of the following, is three times lower than previous.So , the next few members will be $3:3=1,......1:3=\frac{1}{3},.......\frac{1}{3}:3=\frac{1}{9},...$

Geometric progression(geometric sequence) is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed non-zero number called the ***common ratio***.

As we see, geometric progression may be increasing or decreasing.

In example 1. common ratio $q=2$ (increas)

In example 2. common ratio $q=\frac{1}{3}$ (decreas)

The first member of geometric progression is marked with b_1, and member on n-th place is b_n.

$$\boxed{b_n=b_1\cdot q^{n-1}}$$

The sum of the terms of a geometric progression is known as a **geometric series**.

i) for $q>1$

$$S_n=\frac{b_1(q^n-1)}{q-1}$$

ii) for $q<1$

$$S_n=\frac{b_1(1-q^n)}{1-q}$$

For this progression is:

$$b_n=\sqrt{b_{n-1}*b_{n+1}}$$

EXAMPLES:

1. Find the geometric progression if : $b_1 + b_3 = 15 \wedge b_2 + b_4 = 30$

Solution:

$$b_1 + b_3 = 15$$
$$b_2 + b_4 = 30$$

Use : $b_n = b_1 * q^{n-1}$

$$b_3 = b_1 * q^2$$
$$b_2 = b_1 * q$$
$$b_4 = b_1 * q^3$$

Replace this in system :

$$b_1 + b_1 * q = 15$$
$$b_1 q + b_1 q^3 = 30$$

$$b_1(1+q^2) = 15$$
$$b_1 q(1+q^2) = 30$$

Here is a "trick" to devide equations.

$$\frac{b_1(1+q^2)}{b_1 q(1+q^2)} = \frac{15}{30} \rightarrow \frac{\cancel{b_1}\ \cancel{(1+q^2)}}{\cancel{b_1}\ q\ \cancel{(1+q^2)}} = \frac{15}{30} \rightarrow \text{Simplify…}$$

$$\frac{1}{q} = \frac{1}{2} \Rightarrow q = 2$$

Let's go back to one of the equation (of course, choose easy one).

$$b_1(1+q^2) = 15$$
$$b_1(1+4) = 15 \Rightarrow b_1 = 3$$

So: 3, 6, 12, 24, 48,… is solution

2. **Calculate the tenth member of geometric progression 1, 3, 9, 27...**

Solution:

$$\underset{b_1}{\underset{\downarrow}{1}},\underset{b_2}{\underset{\downarrow}{3}},\underset{b_3}{\underset{\downarrow}{9}},\underset{b_4}{\underset{\downarrow}{27}},...$$ From here we can conclude that: $b_1 = 1$ and $q = 3$

Next, we will use formula: $b_n = b_1 * q^{n-1}$

$$b_n = b_1 * q^{n-1}$$

$$b_{10} = b_1 * q^{10-1}$$
$$b_{10} = b_1 * q^9$$
$$b_{10} = 1 * 3^9$$
$$b_{10} = 3^9$$
$$b_{10} = 19683$$

3. **In geometric progression we have:** $b_6 - b_4 = 216 \wedge b_3 - b_1 = 8 \wedge S_n = 40$ **Find** b_1, q and n.

Solution:

$$b_6 - b_4 = 216$$
$$b_3 - b_1 = 8$$
$$S_n = 40$$
$$b_n = b_1 * q^{n-1}$$

$$b_6 = b_1 * q^5$$
$$b_4 = b_1 * q^3$$
$$b_3 = b_1 * q^2$$

Replace in the first two equations!

$$\left.\begin{aligned} b_1 \cdot q^5 - b_1 \cdot q^3 = 216 \\ b_1 q^2 - b_1 = 8 \end{aligned}\right\}$$ common…

$$\left.\begin{aligned} b_1 q^3 (q^2 - 1) = 216 \\ b_1 (q^2 - 1) = 8 \end{aligned}\right\}$$ divide them

$$\frac{b_1 q^3 (q^2 - 1)}{b_1 (q^2 - 1)} = \frac{216}{8} \rightarrow \frac{\cancel{b_1} q^3 \cancel{(q^2 - 1)}}{\cancel{b_1} \cancel{(q^2 - 1)}} = \frac{216}{8}$$

$$q^3 = 27 \Rightarrow q^3 = 3^3 \Rightarrow q = 3$$
$$b_1 (q^2 - 1) = 8$$
$$b_1 (3^2 - 1) = 8 \Rightarrow b_1 \cdot 8 = 8 \Rightarrow b_1 = 1$$

$$\frac{1\cdot(3^n-1)}{3-1}=40$$

$$\frac{3^n-1}{2}=40$$

Because $q=3>1$, we will use formula: $S_n=\dfrac{b_1(q^n-1)}{q-1}$ $\Rightarrow 3^n-1=80$

$$3^n=81$$

$$3^n=3^4\Rightarrow n=4$$

4. We have three numbers $b_{1,}b_2$ and b_3 (members of geometric progression) and for them is : $b_1+b_2+b_3=26$

If you add them 1, 6 and 3 they become members of arithmetical progression. Find $b_{1,}b_2$ and b_3.

Solution:

$b_{1,}b_2$, b_3 and $b_1+b_2+b_3=26$ we know that $b_2=b_1q\wedge b_3=b_1q^2$ then

$b_1+b_1q+b_1q^2=26 \longrightarrow \boxed{b_1(1+q+q^2)=26}$

If we add them 1, 6 and 3 we will receive:

$$a_1=b_1+1$$
$$a_2=b_2+6=b_1q+6$$
$$a_3=b_3+3=b_1q^2+3$$

Because they are members of arithmetical progression, must be: $a_2=\dfrac{a_1+a_3}{2}$

$$a_1+a_3=2a_2$$
$$(b_1+1)+(b_1q^2+3)=2(b_1q+6)\rightarrow \text{simplify}\ldots$$
$$b_1+1+b_1q^2+3=2b_1q+12$$
$$b_1q^2-2b_1q+b_1=12-1-3$$
$$\boxed{b_1(q^2-2q+1)=8}$$

Now, we can make system:

$$\left.\begin{array}{l} b_1(q^2+q+1)=26 \\ b_1(q^2-2q+1)=8 \end{array}\right\} \text{ divide them}$$

$$\frac{q^2+q+1}{q^2-2q+1}=\frac{26}{8}$$

$$26(q^2-2q+1)=8(q^2+q+1)\,/:2$$

$$13(q^2-2q+1)=4(q^2+q+1)$$

$$13q^2-26q+13=4q^2+4q+4$$

$$9q^2-30q+9=0$$

$$3q^2-10q+3=0 \rightarrow \text{ square equation “by q”}$$

$$q_{1,2}=\frac{10\pm 8}{3\cdot 2}=\frac{10\pm 8}{6}$$

$$q_1=3 \wedge q_2=\frac{1}{3}$$

$$q=3$$

$$b_1=\frac{26}{q^2+q+1}=\frac{26}{13}=2$$

or

$$q=\frac{1}{3}$$

$$b_1=\frac{26}{\frac{1}{9}+\frac{1}{3}+1}=\frac{26}{\frac{13}{9}}=18$$

Solution (for q = 3):

2, 6, 18,… geometric progression

3, 12, 21,… arithmetical progression

Solution (for q = 1/3)

18, 6, 2 … geometric progression

19, 12, 5… arithmetical progression

5. Calculate the sum of n numbers in form 1, 11, 111, 1111, 11111,……..

Solution:

"Trick" is to write numbers in different manner:

$$1=\frac{10-1}{9}$$

$$11=\frac{100-1}{9}=\frac{10^2-1}{9}$$

$$111=\frac{1000-1}{9}=\frac{10^3-1}{9}$$

…………………………….

So:

$$S_n=1+11+111+\ldots=$$

$$=\frac{10-1}{9}+\frac{10^2-1}{9}+\frac{10^3-1}{9}+\ldots+\frac{10^n-1}{9}$$

$$=\frac{1}{9}\left[10-1+10^2-1+10^3-1+\ldots+10^n-1\right]$$

$$=\frac{1}{9}\left[\underbrace{10+10^2+\ldots+10^n}-n\right]$$

geometric progression $\rightarrow$ $b_1=10 \wedge q=10$ and $S_n=\frac{b_1(q^n-1)}{q-1}$

$$S_n=\frac{1}{9}\left[\frac{10\cdot(10^n-1)}{10-1}-n\right]$$

$$S_n=\frac{1}{9}\left[\frac{10(10^n-1)}{9}-n\right]=\frac{1}{81}\left[10(10^n-1)-9n\right]$$

6. Calculate the sum of n numbers in form $\frac{5}{6}, \frac{11}{12}, \frac{23}{24}, \frac{47}{48}, \dots\dots$

Solution:

"Trick" is, same as in the previous task, to write numbers in different manner:

$$\frac{5}{6} = \frac{6-1}{6} = 1 - \frac{1}{6}$$
$$\frac{11}{12} = \frac{12-1}{12} = 1 - \frac{1}{12}$$
$$\frac{23}{24} = \frac{24-1}{24} = 1 - \frac{1}{24}$$

………………………..

$$S_n = \frac{5}{6} + \frac{11}{12} + \frac{23}{24} + \dots = 1 - \frac{1}{6} + 1 - \frac{1}{12} + 1 - \frac{1}{24} + \dots = n - \underbrace{\left(\frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \dots\right)}$$

geometric progression

$$b_1 = \frac{1}{6} \qquad q = \frac{1}{2}$$
$$S_{n^*} = \frac{b_1(1-q^n)}{1-q}$$
$$S_{n^*} = \frac{\frac{1}{6}(1-(\frac{1}{2})^n)}{1-\frac{1}{2}}$$
$$S_{n^*} = \frac{1}{3}(1-(\frac{1}{2})^n)$$

So:

$$S_n = n - S_{n^*}$$
$$S_n = n - \frac{1}{3}\left[1-(\frac{1}{2})^n\right] \quad \text{is solution}$$

Infinite geometric series

If we have real numbers $a_1, a_2, ..., a_n, ...$

Form $a_1 + a_2 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n$ is called infinite series

For $a, aq, aq^2, ..., aq^n, ...$ is $a(1 + q + q^2 + ... + q^n + ...) = a\sum_{n=0}^{\infty} q^n$

$S = \frac{a}{1-q}$ for $|q| < 1$

7. Decimal number 0,7777777…write in form of fraction

Solution:

$$0,7777... = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + ...$$
$$= \frac{7}{10}(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + ...)$$
$$= \frac{7}{10}\underbrace{(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + ...)}$$

geometric series : $a = \frac{7}{10}, q = \frac{1}{10}$

$$S = \frac{a}{1-q} = \frac{\frac{7}{10}}{1 - \frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{9}$$

So: **0,7777777…** = $\frac{7}{9}$

8. **Decimal number** 0,3444…. **write in form of fraction**

Solution:

$$0,3444... = \frac{3}{10} + \frac{4}{100} + \frac{4}{1000} + \frac{4}{10000} + ...$$

$$= \frac{3}{10} + \frac{4}{100} \cdot \underbrace{(1 + \frac{1}{10} + \frac{1}{100} + ...)}$$

$$a = \frac{4}{100}, q = \frac{1}{10}$$

$$S = \frac{3}{10} + \frac{\frac{4}{100}}{1 - \frac{1}{10}}$$

$$= \frac{3}{10} + \frac{\frac{4}{100}}{\frac{9}{10}}$$

$$= \frac{3}{10} + \frac{4}{90} = \frac{31}{90}$$

So: 0,3444….= $\boxed{\frac{31}{90}}$

9. **Decimal number** 2,717171…. **write in form of fraction.**

Solution:

$$2{,}717171... = 2 + \frac{7}{10} + \frac{1}{100} + \frac{7}{1000} + \frac{1}{10000} + ...$$

Here we see 2 geometric series:

$$\frac{7}{10} + \frac{7}{1000} + \frac{7}{100000} + ... = \frac{7}{10}(1 + \frac{1}{100} + ...)$$
$$\frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + ... = \frac{1}{100}(1 + \frac{1}{100} + ...)$$

$$S_1 = \frac{\frac{7}{10}}{1 - \frac{1}{100}} = \frac{\frac{7}{10}}{\frac{99}{100}} = \frac{70}{99}$$
$$S_2 = \frac{\frac{1}{100}}{1 - \frac{1}{100}} = \frac{\frac{1}{100}}{\frac{99}{100}} = \frac{1}{99}$$

Let's go back to the task:

$2{,}717171... = 2 + \frac{70}{99} + \frac{1}{99} = \frac{269}{99}$ is solution