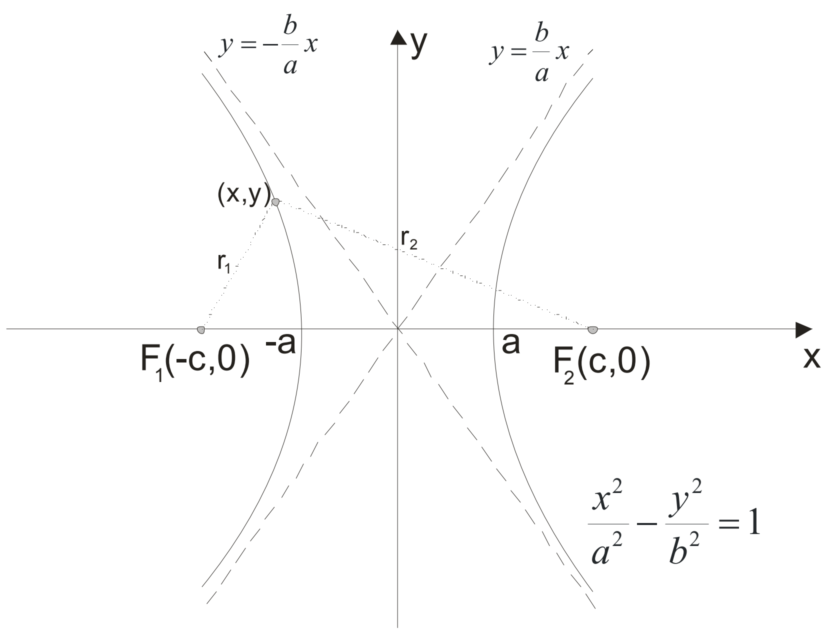
Hyperbola

Hyperbola is the set of points in the plane with the attribute the difference to the distance of any point of the two given points of a constant number.



a- is a real semi-axis (2a is the real axis)

b- is the imaginary semi-axis (the imaginary axis is 2b)

r_1, r_2 are radius vectors and for them applies: $|r_1 - r_2| = 2a$

$F_1(-c,0), F_2(c,0)$ are gisela and $c^2 = a^2 + b^2$

$e = \frac{c}{a}$ $\rightarrow$ the eccentricity (e >1)

Lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are the hyperbola asymptotes

The main equation is $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ or $b^2x^2 - a^2y^2 = a^2b^2$

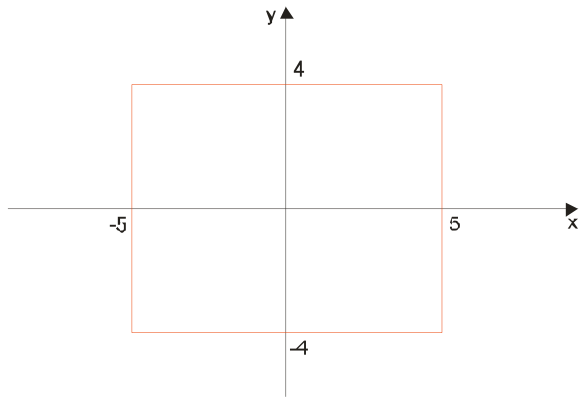
How to draw a hyperbola?

For example : $\frac{x^2}{25} - \frac{y^2}{16} = 1$.

Comparing with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ conclude that $a^2 = 25$ and $b^2 = 16$

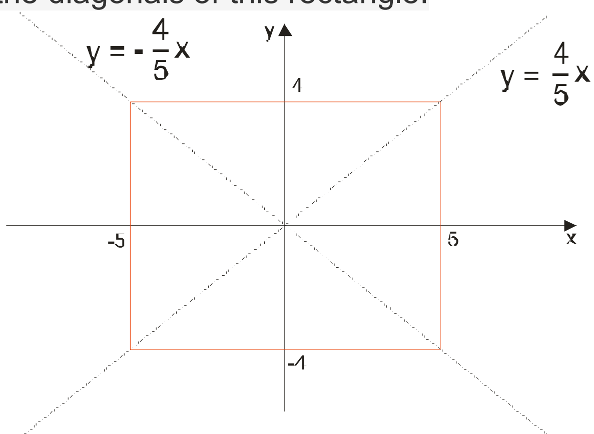
From here is $a = \pm 5$ and $b = \pm 4$

Draw a rectangle :

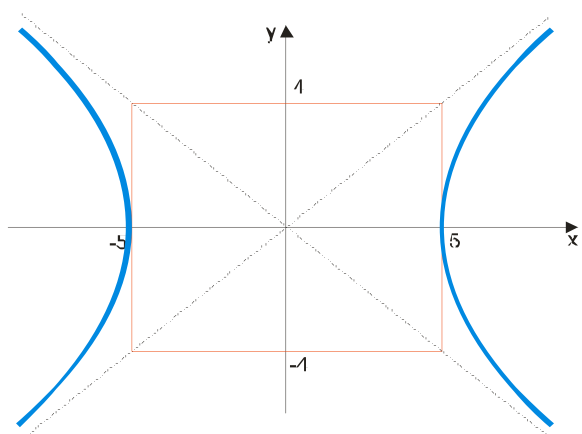


Asymptotes are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$,for our example $y = \frac{4}{5}x$ and $y = -\frac{4}{5}x$.

The graph asymptotes contain the diagonals of this rectangle:



Now draw the hyperbola:



Example 1.

Determine the equation of hyperbola if its semi-axis ratio is 3 : 4 and c = 15

Solution

Use the "trick with k"

$b:a=3:4$

$b=3k$ and $c^2=a^2+b^2$ so:

$a=4k$

$c^2=a^2+b^2$

$c^2=(4k)^2+(3k)^2$

$15^2=16k^2+9k^2$

$225=25k^2$

$k^2=\frac{225}{25}$

$k^2=9$

$k=3$

Let's go back to find a and b:

$b=3k=3\cdot 3=9\rightarrow b^2=81$

$a=4k=4\cdot 3=12\rightarrow a^2=144$

solution:

$$\boxed{\frac{x^2}{144}-\frac{y^2}{81}=1}$$

Example 2.

Determine the equation of hyperbola if the distance between the focus is $10\sqrt{2}$, *and the equations of its asymptotes are* $y=\pm\frac{3}{4}x$

Solution

the distance between the focus is $2c=10\sqrt{2}$ so $c=5\sqrt{2}$.

$y=\pm\frac{3}{4}x$ compared with $y=\pm\frac{b}{a}x$ and we get $\frac{b}{a}=\frac{3}{4}\rightarrow b=\frac{3}{4}a$

This substitute in $c^2=a^2+b^2$

$(5\sqrt{2})^2 = a^2 + (\frac{3}{4}a)^2$

$50 = a^2 + \frac{9}{16}a^2$

$50 = \frac{25}{16}a^2$

$a^2 = 32$

then $b^2 = c^2 - a^2$, $b^2 = 50 - 32$, $b^2 = 18$ and solution is $\boxed{\frac{x^2}{32} - \frac{y^2}{18} = 1}$

<u>Hyperbola and line</u>

Similarly as in the circle and ellipse , to determine the mutual position of line and hyperbola, solve the system of equations:

$y = kx + n$ and $b^2x^2 - a^2y^2 = a^2b^2$

- If the system has no solution, then the line and the hyperbola is not cut, that is $a^2k^2 - b^2 < n^2$
- If the system has two solutions, then line cut hyperbola in two points $a^2k^2 - b^2 > n^2$
- If the system has one solution, line is tangent, and satisfies the **contact condition:**

$$a^2k^2 - b^2 = n^2$$

Note

If we seek an tangent line at a given point (x_0, y_0) which belongs to the hyperbola, we have formula:

$$t: \frac{x \cdot x_0}{a^2} - \frac{y \cdot y_0}{b^2} = 1$$

<u>Example 3.</u>

Write the equation of tangent hyperbola $x^2 - y^2 = 40$ in point M(x,9) belonging hyperbole.

<u>Solution</u>

First, we determine the coordinate x in point M(x,9).

$x^2 - y^2 = 40$

$x^2 - 9^2 = 40$

$x^2 = 40 + 81$

$x^2 = 141$

$x = 11 \vee x = -11$

So we have two points that satisfy : (-11,9) and (11,9)

$$x^2 - y^2 = 40 \quad ..../:40$$

$$\frac{x^2}{40} - \frac{y^2}{40} = 1$$

Use: $t: \dfrac{x \cdot x_0}{a^2} - \dfrac{y \cdot y_0}{b^2} = 1$

$$t_1 : \frac{x \cdot (-11)}{40} - \frac{y \cdot 9}{40} = 1 \qquad t_2 : \frac{x \cdot 11}{40} - \frac{y \cdot 9}{40} = 1$$

$$t_1 : -11x - 9y = 40 \qquad t_2 : 11x - 9y = 40$$

$$t_1 : -11x - 9y - 40 = 0 \qquad t_2 : 11x - 9y - 40 = 0$$

Example 4.

Write the equation of hyperbola if known equation of its tangent line: 5x-7y-1=0 and x-y-1=0

Solution

Tangent must satisfy the contact condition : $a^2k^2 - b^2 = n^2$.

Shift tangent line in the explicit form:

$$5x - 7y - 1 = 0$$
$$-7y = -5x + 1......./:(-7)$$
$$y = \frac{5}{7}x - \frac{1}{7}$$
$$k = \frac{5}{7}$$
$$n = -\frac{1}{7}$$

and

$$x - y - 1 = 0$$
$$-y = -x + 1$$
$$y = x - 1$$
$$k = 1$$
$$n = -1$$

$$a^2k^2 - b^2 = n^2$$
$$a^2(\frac{5}{7})^2 - b^2 = (-\frac{1}{7})^2$$
$$a^2\frac{25}{49} - b^2 = \frac{1}{49}$$
$$25a^2 - 49b^2 = 1$$

and

$$a^2k^2 - b^2 = n^2$$
$$a^2 1^2 - b^2 = (-1)^2$$
$$a^2 - b^2 = 1$$

Now create a system:

$a^2 - b^2 = 1$

$\underline{25a^2 - 49b^2 = 1}$

$a^2 = b^2 + 1$

$25(b^2+1) - 49b^2 = 1$ and solution is: $\boxed{\frac{x^2}{2} - \frac{y^2}{1} = 1}$

$25b^2 + 25 - 49b^2 = 1$

$-24b^2 = -24$

$b^2 = 1 \to a^2 = b^2 + 1 \to a^2 = 2$

Example 5.

Determine the angle under which the curves intersect $3x^2 + 4y^2 = 84$ *and* $3x^2 - 4y^2 = 12$.

Solution

First, find the intersection point by solving the system of equations:

$3x^2 + 4y^2 = 84$

$\underline{3x^2 - 4y^2 = 12}$

$6x^2 = 96$

$x^2 = 16$

$x_1 = 4 \to 3 \cdot 4^2 + 4y^2 = 84 \to 4y^2 = 84 - 48 \to 4y^2 = 16 \to y^2 = 4 \to y = \pm 2$

$x_2 = -4 \to 3 \cdot (-4)^2 + 4y^2 = 84 \to 4y^2 = 84 - 48 \to 4y^2 = 16 \to y^2 = 4 \to y = \pm 2$

Cut in:

$(4,2); (4,-2); (-4,2); (-4,-2)$

Angle under which the curves intersect is the angle between tangents in one of the points of intersection!

We will take the point (4,2) and it set the tangent to the ellipse and hyperbola

$t_e : \frac{x \cdot x_0}{a^2} + \frac{y \cdot y_0}{b^2} = 1$ and $t_h : \frac{x \cdot x_0}{a^2} - \frac{y \cdot y_0}{b^2} = 1$

$3x^2 + 4y^2 = 84$ ..../:84

$\frac{3x^2}{84} + \frac{4y^2}{84} = 1$

$\frac{x^2}{28} + \frac{y^2}{21} = 1$

$\frac{x \cdot 4}{28} + \frac{y \cdot 3}{21} = 1$

$\frac{x}{7} + \frac{y}{7} = 1$

$x + y = 7$

$y = -x + 7$

$k_1 = -1$

$3x^2 - 4y^2 = 12$................../:12

$\frac{3x^2}{12} - \frac{4y^2}{12} = 1$

$\frac{x^2}{4} - \frac{y^2}{3} = 1$

$\frac{x \cdot 4}{4} - \frac{y \cdot 3}{3} = 1$

$x - y = 1$

$y = x - 1$

$k_2 = 1$

We can use the formula for the angle between two lines: $tg\alpha = \left|\frac{k_2 - k_1}{1 + k_1 \cdot k_2}\right|$, but we can immediately conclude that the cut angle is 90^0.

How?

Well we know that the condition of normality is $k_1 \cdot k_2 = -1$, and this is obviously satisfied!