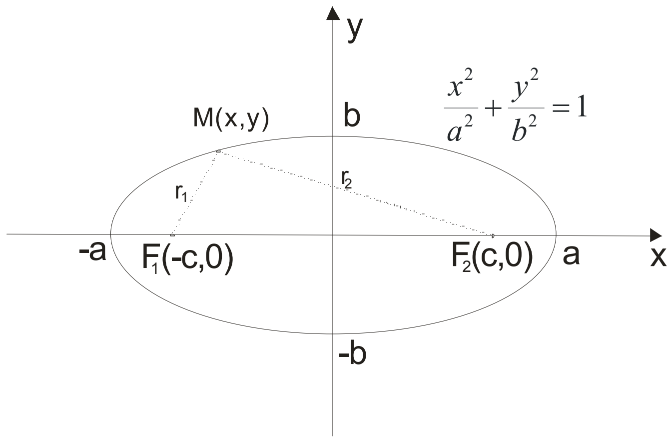
ELLIPSE

Ellipse is the set of points in the plane with the feature that the sum of the distance of any point of the two given points (focus) a constant number.



r_1, r_2 are the radius vectors and for each point is $r_1 + r_2 = 2a$ (constant number)

$F_1(-c,0)$ is focus and $F_2(c,0)$ is focus , where is $c^2 = a^2 - b^2$

a – is large semi-axis and 2a is the major axis

b –is small semi-axis and 2b is the minor axis

$e = \frac{c}{a}$ is eccentricity (e <1)

Main ellipse equation is : $\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ or $b^2x^2 + a^2y^2 = a^2b^2$

Example 1.

Determine the equation of an ellipse, if fokus has coordinates $(\pm 3, 0)$ *and length of large axis is 12.*

Solution:

c = 3.
Since 2a = 12 then a = 6. ($a^2 = 36$)

Use $c^2 = a^2 - b^2$ to find b:

$c^2 = a^2 - b^2$

$3^2 = 6^2 - b^2$

$9 = 36 - b^2$ Substitution in the ellipse equation and $\frac{x^2}{36}+\frac{y^2}{27}=1$ here's the solution.

$b^2 = 36 - 9$

$b^2 = 27$

Example 2.

Determine the equation of an ellipse containing the point M(6,4) and N (- 8, 3)

Solution:

Coordinates of given point we will replace in the ellipse equation, but it is better to use a form $b^2x^2 + a^2y^2 = a^2b^2$

$$M(6,4) \rightarrow b^2x^2 + a^2y^2 = a^2b^2$$
$$b^2 6^2 + a^2 4^2 = a^2b^2$$
$$36b^2 + 16a^2 = a^2b^2$$

$$N(-8,3) \rightarrow b^2x^2 + a^2y^2 = a^2b^2$$
$$b^2(-8)^2 + a^2 3^2 = a^2b^2$$
$$64b^2 + 9a^2 = a^2b^2$$

comparing the left side of equality, we get:

$$36b^2 + 16a^2 = 64b^2 + 9a^2$$
$$16a^2 - 9a^2 = 64b^2 - 36b^2$$
$$7a^2 = 28b^2$$
$$a^2 = 4b^2$$

Now go back to one of the two equal and replace the resulting value:

$$a^2 = 4b^2$$
$$\underline{64b^2 + 9a^2 = a^2b^2}$$
$$64b^2 + 9 \cdot 4b^2 = 4b^2b^2$$
$$100b^2 = 4b^4$$
$$4b^4 - 100b^2 = 0$$
$$4b^2(b^2 - 25) = 0$$
$$b^2 = 25 \rightarrow a^2 = 4 \cdot 25 \rightarrow a^2 = 100$$

Substituting this into equation, we get solution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$

Ellipse and line

Similarly as in the circle, to determine the mutual position of line and ellipses, solve the system of equations:

$y = kx + n$ and $b^2x^2 + a^2y^2 = a^2b^2$

- If the system has no solution, then the line and the ellipse is not cut, that is $a^2k^2 + b^2 < n^2$
- If the system has two solutions, then line cut ellipse in two points $a^2k^2 + b^2 > n^2$
- If the system has one solution, line is tangent, and satisfies the **contact condition:**

$$a^2k^2 + b^2 = n^2$$

Note

If we seek an ellipse tangent line at a given point (x_0, y_0) which belongs to the ellipse, we have formula:

$$t: \frac{x \cdot x_0}{a^2} + \frac{y \cdot y_0}{b^2} = 1$$

Example 3.

In intersecting points ellipse $x^2 + 3y^2 = 28$ *and line* $5x - 3y - 14 = 0$ *were constructed tangent. Find them.*

Solution:

$5x - 3y - 14 = 0$ find x from here

$x^2 + 3y^2 = 28$

$x = \frac{3y+14}{5} \rightarrow$ replace in second equation

$(\frac{3y+14}{5})^2 + 3y^2 = 28$

$\frac{9y^2 + 84y + 196}{25} + 3y^2 = 28$

$9y^2 + 84y + 196 + 75y^2 = 700$

$84y^2 + 84y - 504 = 0 .... / :84$

$y^2 + y - 6 = 0$

$y_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y_1 = 2$

$y_2 = -3$

then

$y_1 = 2 \rightarrow x_1 = \frac{3 \cdot 2 + 14}{5} \rightarrow x_1 = 4$

$y_2 = -3 \rightarrow x_2 = \frac{3 \cdot (-3) + 14}{5} \rightarrow x_2 = 1$

We got the points of intersection: (4, 2) and (1,-3).

Since the points are **on the ellipse** we will use formula: $t:\frac{x\cdot x_0}{a^2}+\frac{y\cdot y_0}{b^2}=1$

First, the ellipse move in another form:

$$x^2+3y^2=28 \quad ..../:28$$

$$\frac{x^2}{28}+\frac{3y^2}{28}=1$$

$$\frac{x^2}{28}+\frac{y^2}{\frac{28}{3}}=1$$

$$for \quad (4,2)\rightarrow t_1:\frac{x\cdot 4}{28}+\frac{y\cdot 2}{\frac{28}{3}}=1$$

$$t_1: 2x+3y-14=0$$

$$for \quad (1,-3)\rightarrow t_2:\frac{x\cdot 1}{28}+\frac{y\cdot(-3)}{\frac{28}{3}}=1$$

$$t_2: x-9y-28=0$$

Example 4.

Determine the parameter p so that the line $y + x + p = 0$ represents the tangent line ellipses $2x^2+3y^2=30$

Solution

Here we have to used the contact condition . First, arrange an ellipse and line .From them we read what we need…

$$y+x+p=0$$

$$y=-x-p$$

From here we have $k=-1$ and $n=-p$

$$2x^2+3y^2=30 \quad ..../:30$$

$$\frac{2x^2}{30}+\frac{3y^2}{30}=1$$

$$\frac{x^2}{15}+\frac{y^2}{10}=1\rightarrow a^2=15 \quad \text{and} \quad b^2=10$$

contact condition:

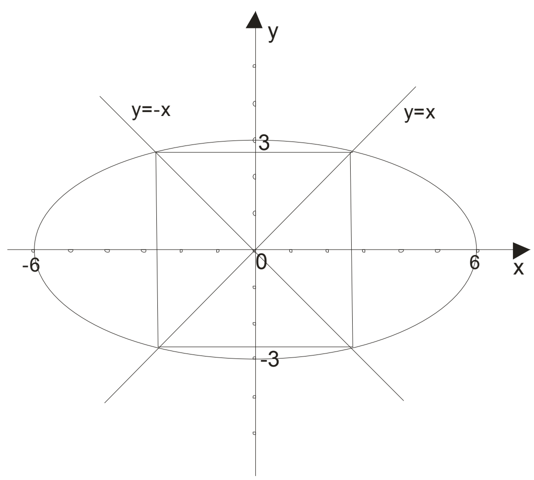
$$a^2k^2 + b^2 = n^2$$
$$15(-1)^2 + 10 = (-p)^2$$
$$25 = p^2$$
$$p_1 = 5$$
$$p_2 = -5$$

Example 5.

The square is inscribed in an ellipse $x^2 + 4y^2 = 36$. *Calculated area of the square.*

Solution

Here it is necessary to draw a picture and set the problem…



What can we see?

Line y = x and y = - x in the intersection of the ellipse given vertices of the inscribed square!

So, solve the system y = x and $x^2 + 4y^2 = 36$

$$x^2 + 4y^2 = 36$$
$$\underline{y = x}$$
$$x^2 + 4x^2 = 36$$
$$5x^2 = 36$$
$$x^2 = \frac{36}{5} \rightarrow x_1 = \frac{6}{\sqrt{5}}, x_2 = -\frac{6}{\sqrt{5}}$$

As y = x and y = - x we have that coordinates of the vertices are:

$$A(\frac{6}{\sqrt{5}},\frac{6}{\sqrt{5}}); B(\frac{6}{\sqrt{5}},-\frac{6}{\sqrt{5}}); C(-\frac{6}{\sqrt{5}},-\frac{6}{\sqrt{5}}); D(-\frac{6}{\sqrt{5}},\frac{6}{\sqrt{5}})$$

Mark squares page with *a*. Its length, we get as the distance between points A and B ,for example.

$$d(A,B)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$a=\sqrt{(\frac{6}{\sqrt{5}}-\frac{6}{\sqrt{5}})^2+(-\frac{6}{\sqrt{5}}-\frac{6}{\sqrt{5}})^2}$$

$$a=\sqrt{0+(-\frac{12}{\sqrt{5}})^2}$$

$$a^2=\frac{144}{5}$$

We know that the area of squares calculated by the formula $A_{\square}=a^2$

$$\boxed{A_{\square}=\frac{144}{5}}$$