Primer 4.

Funkciju $f(x)=|x|-1$ razviti u Furijeov red na segmentu $[-1,1]$ a zatim izračunati sumu reda $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.

Rešenje:

Kako je $f(-x)=|-x|-1=|x|-1=f(x)$ zaključujemo da je funkcija parna .

Koristimo formule:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l})$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx \qquad a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx \qquad b_n=0$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx=\frac{1}{1}\int_{-1}^{1}(x-1)dx=2\int_{0}^{1}(x-1)dx=2\left(\frac{x^2}{2}-x\right)\Big/_0^1=-1$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx=\frac{1}{1}\int_{-1}^{1}(x-1)\cos\frac{n\pi x}{1}dx=2\int_{0}^{1}(x-1)\cos n\pi x dx$$

Kao i uvek , ovaj integral ćemo rešiti na stranu uz pomoć parcijalne integracije:

$$\int(x-1)\cos n\pi x dx=\begin{vmatrix}x-1=u & \cos n\pi x dx=dv\\ dx=du & \frac{1}{n\pi}\sin n\pi x=v\end{vmatrix}=(x-1)\cdot\frac{1}{n\pi}\sin n\pi x-\int\frac{1}{n\pi}\sin n\pi x dx=$$

$$=\frac{(x-1)\sin n\pi x}{n\pi}-\frac{1}{n\pi}\int\sin n\pi x dx==\frac{(x-1)\sin n\pi x}{n\pi}+\frac{1}{n\pi}\frac{1}{n\pi}\cos n\pi x$$

$$=\frac{(x-1)\sin n\pi x}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi x$$

Sad se vratimo da ubacimo granice:

$$a_n=2\int_{0}^{1}(x-1)\cos n\pi x dx=2\left(\frac{(x-1)\sin n\pi x}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi x\right)\Big/_0^1=$$

$$=2\left[\left(\frac{(1-1)\sin n\pi 1}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi 1\right)-\left(\frac{(0-1)\sin n\pi 0}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi 0\right)\right]$$

$$=2\left(\frac{1}{(n\pi)^2}\cos n\pi-\frac{1}{(n\pi)^2}\right)=\frac{2}{(n\pi)^2}\left(\cos n\pi-1\right)=\frac{2}{(n\pi)^2}\left((-1)^n-1\right)$$

Slično kao u prethodnim primerima, razmišljamo o parnim i neparnim *n,* pa je:

$$a_n = \begin{cases} 0, & n = 2k \\ -\dfrac{4}{(n\pi)^2}, & n = 2k-1 \end{cases}$$

Sad idemo u početnu formulu:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$$

$$f(x) = |x| - 1 = \frac{1}{2}(-1) + \sum_{k=1}^{\infty}\frac{-4}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi x}{1}$$

$$\boxed{|x| - 1 = -\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\pi x}{(2k-1)^2}}$$

Pogledajmo i sumu koja se traži: $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$. Vidimo da u našem redu treba ubaciti x = 0 :

$$|0| - 1 = -\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\pi 0}{(2k-1)^2}$$

$$-1 = -\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

$$\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{1}{2} \to \boxed{\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}}$$

Primer 5.

Funkciju *f(x) = x-2* razviti u Furijeov red na segmentu *[1,3]* .

Rešenje:

Moramo koristiti formule:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a})$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)dx \qquad a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}dx \qquad b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}dx$$

Dakle, imamo:

$$a_0=\frac{2}{3-1}\int_1^3(x-2)dx=\left(\frac{x^2}{2}-2x\right)/_1^3=\left(\frac{3^2}{2}-6\right)-\left(\frac{1^2}{2}-2\right)=0$$

$$a_n=\frac{2}{3-1}\int_1^3(x-2)\cos\frac{2n\pi x}{3-1}dx=$$

$$=\int_1^3(x-2)\cos n\pi x dx$$

Da rešimo najpre ovo bez granica:

$$\int(x-2)\cos n\pi x dx=\begin{vmatrix}x-2=u & \cos n\pi x dx=dv\\ dx=du & \frac{1}{n\pi}\sin n\pi x=v\end{vmatrix}=(x-2)\cdot\frac{1}{n\pi}\sin n\pi x-\int\frac{1}{n\pi}\sin n\pi x dx=$$

$$=\frac{(x-2)\sin n\pi x}{n\pi}-\frac{1}{n\pi}\int\sin n\pi x dx==\frac{(x-2)\sin n\pi x}{n\pi}+\frac{1}{n\pi}\frac{1}{n\pi}\cos n\pi x$$

$$=\frac{(x-2)\sin n\pi x}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi x$$

$$a_n=\int_1^3(x-2)\cos n\pi x dx=\left(\frac{(x-2)\sin n\pi x}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi x\right)/_1^3=$$

$$=\left(\frac{(3-2)\sin n\pi 3}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi 3\right)-\left(\frac{(1-2)\sin n\pi 1}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi 1\right)=$$

$$=\frac{\sin 3n\pi}{n\pi}+\frac{1}{(n\pi)^2}\cos 3n\pi+\frac{\sin n\pi}{n\pi}-\frac{1}{(n\pi)^2}\cos n\pi$$

$$=\frac{1}{(n\pi)^2}\cos 3n\pi-\frac{1}{(n\pi)^2}\cos n\pi$$

$$=\frac{1}{(n\pi)^2}[\cos 3n\pi-\cos n\pi]$$

Sećate se trigonometrijske formulice: $\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$, ako nju upotrebimo:

$$a_n=\frac{1}{(n\pi)^2}[\cos 3n\pi-\cos n\pi]=\frac{1}{(n\pi)^2}[-2\sin 2n\pi\cdot\sin n\pi]=0\rightarrow\boxed{a_n=0}$$

Još da nadjemo:

$$b_n=\frac{2}{3-1}\int_1^3(x-2)\sin\frac{2n\pi x}{3-1}dx=\int_1^3(x-2)\sin n\pi x dx$$

$$\int (x-2)\sin n\pi x dx = \begin{vmatrix} x-2=u & \sin n\pi x dx = dv \\ dx = du & -\frac{1}{n\pi}\cos n\pi x = v \end{vmatrix} = -(x-2)\cdot\frac{1}{n\pi}\cos n\pi x + \int \frac{1}{n\pi}\cos n\pi x dx =$$

$$= \frac{-(x-2)\cos n\pi x}{n\pi} + \frac{1}{n\pi}\int \cos n\pi x dx = = \frac{-(x-2)\cos n\pi x}{n\pi} + \frac{1}{n\pi}\frac{1}{n\pi}\sin n\pi x$$

$$= \frac{-(x-2)\cos n\pi x}{n\pi} + \frac{1}{(n\pi)^2}\sin n\pi x$$

Da ubacimo granice:

$$b_n = \int_1^3 (x-2)\sin n\pi x dx = \left(\frac{-(x-2)\cos n\pi x}{n\pi} + \frac{1}{(n\pi)^2}\sin n\pi x\right) / \begin{matrix} 3 \\ 1 \end{matrix} =$$

$$\left(\frac{-(3-2)\cos n\pi 3}{n\pi} + \frac{1}{(n\pi)^2}\sin n\pi 3\right) - \left(\frac{-(1-2)\cos n\pi x}{n\pi} + \frac{1}{(n\pi)^2}\sin n\pi 1\right) =$$

$$-\frac{\cos n\pi 3}{n\pi} - \frac{\cos n\pi x}{n\pi} = -\frac{1}{n\pi}\left(\cos 3n\pi + \cos n\pi\right)$$

Opet mora formulica: $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$b_n = -\frac{1}{n\pi}\left(\cos 3n\pi + \cos n\pi\right) = -\frac{1}{n\pi}2\underbrace{\boxed{\cos 2n\pi}}_{ovo\ je\ 1}\cos n\pi = -\frac{2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$$

$$\boxed{b_n = (-1)^{n+1}\frac{2}{n\pi}}$$

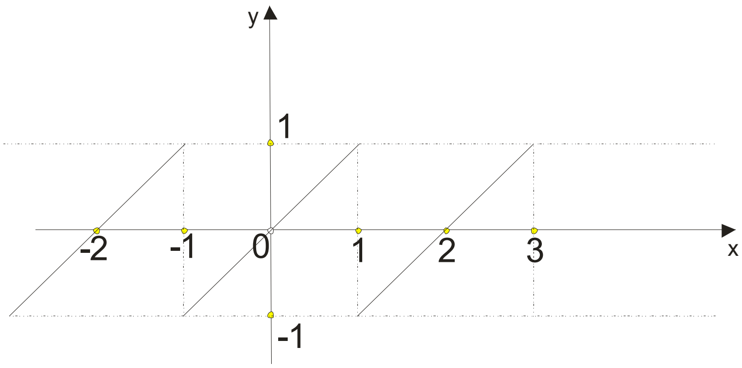
Sad idemo u početnu formulu:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos\frac{2n\pi x}{b-a} + b_n\sin\frac{2n\pi x}{b-a})$$

Pazite:

f(1-0)=1 , f(1+0)=1 i f(3-0)=1, f(3+0)=-1

pogledajte sliku:



pa je

$$S(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n\pi}\sin\frac{2n\pi x}{3-1}$$

$$S(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n\pi}\sin n\pi x=\{\begin{matrix} x-2, & x\in(1,3) \\ 0, & x\in\{1,3\}\end{matrix}$$

Primer 6.

Funkciju $f(x)=\begin{Bmatrix} x, & x\in(0,1) \\ 2-x, & x\in[1,2]\end{Bmatrix}$ razviti u red po:

a) po sinusima

b) po cosinusima

Rešenje:

a)

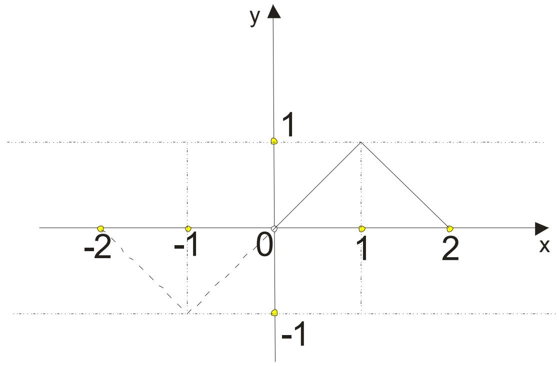
Funkciju $f(x)=\begin{Bmatrix} x, & x\in(0,1) \\ 2-x, & x\in[1,2]\end{Bmatrix}$ razviti u red po sinusima.

Da bi smo razvili ovu funkciju po sinusima, moramo je dodefinisati do **neparne** funkcije.

To ćemo obaviti na sledeći način:

$$F(x)=\begin{Bmatrix} 2-x, & x\in[1,2] \\ x, & x\in(-1,1) \\ -2-x, & x\in[-2,-1]\end{Bmatrix}$$

Pogledajmo kako ova funkcija izgleda na slici:



Naravno da su ovde a_0 i a_n jednaki nuli a tražimo: $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{2}\int_{0}^{2} f(x)\sin\frac{n\pi x}{2}dx = \int_{0}^{2} f(x)\sin\frac{n\pi x}{2}dx$$

Zbog načina na koji je funkcija definisana, ovaj integral rastavljamo na dva:

$$b_n = \int_{0}^{2} f(x)\sin\frac{n\pi x}{2}dx = \int_{0}^{1} x\sin\frac{n\pi x}{2}dx + \int_{1}^{2} (2-x)\sin\frac{n\pi x}{2}dx$$

Nakon rešavanja ovih integrala, metodom parcijalne integracije, na sličan način kao u prethodnim primerima dobijamo:

$$b_n = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Razmišljamo kako se ponaša $\sin\frac{n\pi}{2}$. Znamo da n uzima vrednosti 1,2,3...

Za n = 1 $\sin\frac{n\pi}{2} = \sin\frac{\pi}{2} = 1$

Za n = 2 $\sin\frac{n\pi}{2} = \sin\frac{2\pi}{2} = 0$

Za n = 3 $\sin\frac{n\pi}{2} = \sin\frac{3\pi}{2} = -1$

Za n = 4 $\sin\frac{n\pi}{2} = \sin\frac{4\pi}{2} = 0$

Za n = 5 $\sin\frac{n\pi}{2} = \sin\frac{5\pi}{2} = 1$

Za n = 6 $\sin\frac{n\pi}{2} = \sin\frac{6\pi}{2} = 0$

itd.

Dakle, zaključujemo: $b_n = \begin{cases} 0, & n = 2k \\ (-1)^k \dfrac{8}{(2k+1)^2 \pi^2}, & n = 2k+1 \end{cases}$ k=0,1,2,3…….

Pa je:

$$f(x) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin \frac{(2k+1)\pi x}{2}, za\ x \in (0,2]$$

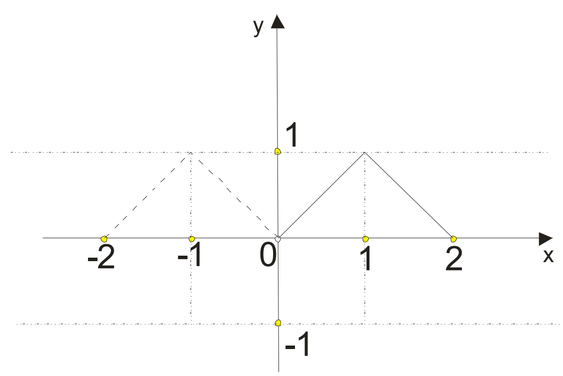
$$F(x) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin \frac{(2k+1)\pi x}{2}, za\ x \in [-2,2]$$

b)

Za razvoj po kosinusima moramo dodefinisati funkciju do parne na sledeći način:

$$F(x) = \begin{Bmatrix} x+2, & x \in [-2,-1] \\ |x|, & x \in (-1,1) \\ x-2, & x \in [1,2] \end{Bmatrix}$$

Data funkcija je prikazana na sledećoj slici:



Naravno, sada je $b_n = 0$ a tražimo: $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$ $\quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{2}{l}\int_{0}^{l} f(x)dx$$

$$a_0 = \frac{2}{2}\int_{0}^{2} f(x)dx = \int_{0}^{2} f(x)dx = \int_{0}^{1} xdx + \int_{1}^{2}(2-x)dx = 1$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_n = \frac{2}{2}\int_{0}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \int_{0}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \int_{0}^{1} x\cos\frac{n\pi x}{2}\,dx + \int_{1}^{2}(2-x)\cos\frac{n\pi x}{2}\,dx$$

Parcijalnom integracijom rešimo ove integrale i dobijamo:

$$a_n = \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}(1+\cos n\pi)$$

Razmislimo kako se ponaša izraz $\cos\frac{n\pi}{2}$ za različite n.

za n=1 $\cos\frac{n\pi}{2} = \cos\frac{\pi}{2} = 0$

za n=2 $\cos\frac{n\pi}{2} = \cos\frac{2\pi}{2} = -1$

za n=3 $\cos\frac{n\pi}{2} = \cos\frac{3\pi}{2} = 0$

za n=4 $\cos\frac{n\pi}{2} = \cos\frac{4\pi}{2} = 1$

itd.

Dakle, ako je *n* neparan broj , *n=2k+1* , tada je $a_n = 0$

Pogledajmo sada parne n, ali oblika *n=4k* ili *n=4k+2* za *k=0,1,2,3…....*

n=4k

$$a_n = \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}(1+\cos n\pi)$$

$$a_{4k} = \frac{8}{(4k)^2\pi^2}\cos\frac{4k\pi}{2} - \frac{4}{(4k)^2\pi^2}(1+\cos 4k\pi) = \frac{8}{16k^2\pi^2}\cos 2k\pi - \frac{4}{16k^2\pi^2}(1+1)$$

$$= \frac{8}{16k^2\pi^2}\cos 2k\pi - \frac{8}{16k^2\pi^2}\cos 2k\pi = 0$$

n=4k+2

$$a_n = \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}(1+\cos n\pi)$$

$$a_{4k} = \frac{8}{(4k+2)^2\pi^2}\cos\frac{(4k+2)\pi}{2} - \frac{4}{(4k+2)^2\pi^2}(1+\cos(4k+2)\pi)$$

$$= \frac{\cancel{8}}{\cancel{4}(2k+1)^2\pi^2}\cos\frac{\cancel{2}(2k+1)\pi}{\cancel{2}} - \frac{\cancel{4}}{\cancel{4}(2k+1)^2\pi^2}(1+\cos 2\pi(2k+1))$$

$$= \frac{2}{(2k+1)^2\pi^2}\underbrace{\cos(2k+1)\pi}_{ovo\ je\ -1} - \frac{1}{(2k+1)^2\pi^2}(1+1)$$

$$= -\frac{2}{(2k+1)^2\pi^2} - \frac{2}{(2k+1)^2\pi^2} = \boxed{-\frac{4}{(2k+1)^2\pi^2}}$$

Konačno imamo:

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^2}, \quad x \in (0,2] \quad \text{i} \quad F(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^2}, \quad x \in [-2,2]$$