POVRŠINSKI INTEGRALI (zadaci- I deo)

1. Izračunati površinski integral $I = \iint_S (6x + 4y + 3z)dS$ **ako je S deo ravni** $x + 2y + 3z = 6$ **, koja pripada prvom oktantu.**

Rešenje:

Važno je još jednom napomenuti da: **POVRŠINSKI INTEGRAL PRVE VRSTE NE ZAVISI OD ORIJENTACIJE KRIVE**

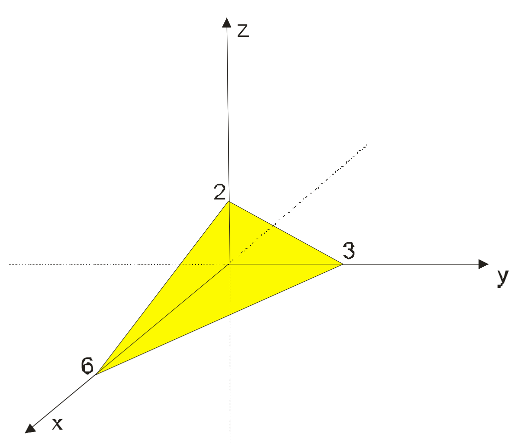
Koristićemo:

ii) Ako jednačina površi S ima oblik z=z(x,y) , gde je z=z(x,y) jednoznačna neprekidno diferencijabilna funkcija, onda je:

$$\iint_S f(x,y,z)ds = \iint_D f[x,y,z(x,y)]\sqrt{1+p^2+q^2}\,dxdy \qquad \text{i}$$

$$p = \frac{\partial z}{\partial x} \quad \text{i} \quad q = \frac{\partial z}{\partial y}$$

Najpre nacrtamo sliku i postavimo problem.



$$x + 2y + 3z = 6......./:6$$
$$\frac{x}{6} + \frac{y}{3} + \frac{z}{2} = 1$$

Segmentni oblik jednačine prave nam daje preseke sa osama (ovo nam sad baš i ne treba al nije rdjavo da pomenemo…)

Naš posao je da izrazimo z iz date jednačine i nadjemo prve parcijalne izvode, odnosno p i q.

$$x + 2y + 3z = 6$$
$$3z = -x - 2y + 6....../:3$$
$$z = -\frac{x}{3} - \frac{2y}{3} + 2$$

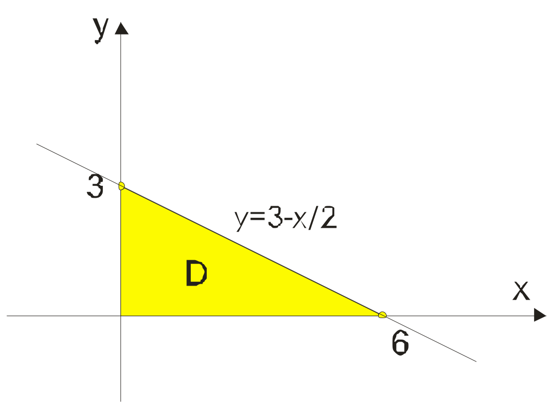
Odavde je:

$$p = \frac{\partial z}{\partial x} = -\frac{1}{3}$$
$$q = \frac{\partial z}{\partial y} = -\frac{2}{3}$$

Spustimo se u ravan z = 0 da odredimo granice.

$$\boxed{z = 0} \to x + 2y + 3\cdot 0 = 6 \to x + 2y = 6 \to \boxed{y = 3 - \frac{x}{2}}$$

Pogledajmo sliku :



$$D: \begin{cases} 0 \le x \le 6 \\ 0 \le y \le 3 - \frac{x}{2} \end{cases}$$

Sad možemo da predjemo na rešavanje integrala:

$$I = \iint_S (6x + 4y + 3z)dS = \iint_D (6x + 4y + \cancel{3} \cdot \frac{6 - x - 2y}{\cancel{3}})\sqrt{1 + p^2 + q^2}\,dxdy$$
$$= \iint_D (6x + 4y + 6 - x - 2y)\sqrt{1 + (-\frac{1}{3})^2 + (-\frac{2}{3})^2}\,dxdy =$$
$$= \iint_D (5x + 2y + 6)\sqrt{\frac{14}{9}}\,dxdy =$$
$$= \frac{\sqrt{14}}{3}\int_0^6 dx \int_0^{3-\frac{x}{2}} (5x + 2y + 6)dy \quad \text{(sredimo sve - podsetite se dvojnih integrala)}$$
$$= \boxed{54\sqrt{14}}$$

2. Rešiti integral $I=\iint_S (x^2+y^2)dS$ ako je S sfera $x^2+y^2+z^2=a^2$.

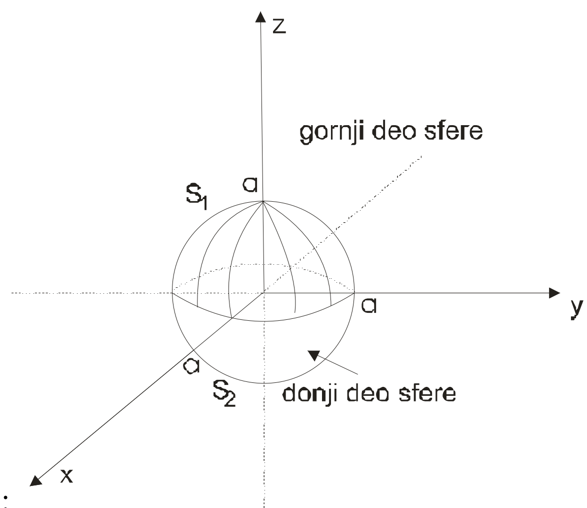
Rešenje:

Najpre moramo izraziti z iz date jednačine:

$$x^2+y^2+z^2=a^2$$
$$z^2=a^2-x^2-y^2$$
$$z=\pm\sqrt{a^2-x^2-y^2}$$

Ovde treba voditi računa da posebno moramo raditi za gornji deo sfere $z=+\sqrt{a^2-x^2-y^2}$ (iznad z = 0 ravni)

i posebno za $z=-\sqrt{a^2-x^2-y^2}$ (ispod z = 0 ravni).



Pogledajmo sliku:

$S_1: z_1=+\sqrt{a^2-x^2-y^2}$ i $S_2: z_2=-\sqrt{a^2-x^2-y^2}$

Za $S_1: z_1=+\sqrt{a^2-x^2-y^2}$ (a slično je i za $S_2: z_2=-\sqrt{a^2-x^2-y^2}$) imamo:

$$p=\frac{\partial z}{\partial x}=\frac{1}{2\sqrt{a^2-x^2-y^2}}\cdot\left(a^2-x^2-y^2\right)'_{po\ x}=\frac{1}{2\sqrt{a^2-x^2-y^2}}\cdot(-2x)=\boxed{\frac{-x}{\sqrt{a^2-x^2-y^2}}}$$

$$q=\frac{\partial z}{\partial y}=\frac{1}{2\sqrt{a^2-x^2-y^2}}\cdot\left(a^2-x^2-y^2\right)'_{po\ y}=\frac{1}{2\sqrt{a^2-x^2-y^2}}\cdot(-2y)=\boxed{\frac{-y}{\sqrt{a^2-x^2-y^2}}}$$

Za $S_2: z_2=-\sqrt{a^2-x^2-y^2}$ će samo biti plusevi….

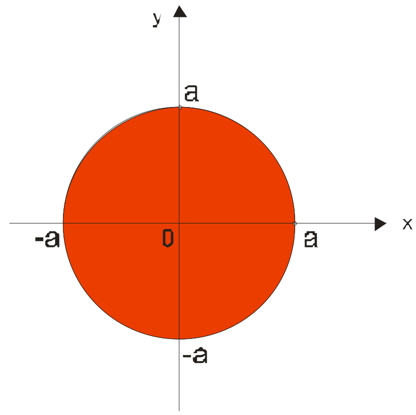
Izračunajmo "na stranu":

$$\sqrt{1+p^2+q^2}=\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{a^2-x^2-y^2}}\right)^2}=\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}=$$

$$=\sqrt{\frac{a^2\cancel{-x^2}\cancel{-y^2}\cancel{+x^2}\cancel{+y^2}}{a^2-x^2-y^2}}=\frac{a}{\sqrt{a^2-x^2-y^2}}$$

Sad spuštamo problem u ravan z = 0:

$$z = 0 \wedge x^2 + y^2 + z^2 = a^2 \to x^2 + y^2 = a^2$$



Oblast D je unutrašnjost ovog kruga! (Ista je i za S_1 i za S_2)

$$I = \iint\limits_S (x^2 + y^2) dS = \iint\limits_{S_1} \; + \iint\limits_{S_2} \; = \iint\limits_D (x^2 + y^2)\sqrt{1 + p^2 + q^2}\, dxdy + \iint\limits_D (x^2 + y^2)\sqrt{1 + p^2 + q^2}\, dxdy =$$

$$= 2\iint\limits_D (x^2 + y^2)\sqrt{1 + p^2 + q^2}\, dxdy = 2\iint\limits_D \frac{a(x^2 + y^2)}{\sqrt{a^2 - x^2 - y^2}} dxdy$$

Uvodimo smene:

$$\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right\} \to x^2 + y^2 = a^2 \to a^2 = r^2 \to r = a \to 0 \le r \le a \wedge 0 \le \varphi \le 2\pi \wedge |J| = r$$

$$I = 2\iint\limits_D \frac{a(x^2 + y^2)}{\sqrt{a^2 - x^2 - y^2}} dxdy = 2\int_0^{2\pi} d\varphi \int_0^a \frac{ar^2}{\sqrt{a^2 - r^2}} rdr = 2\underbrace{\boxed{\int_0^{2\pi} d\varphi}}_{ovo\ je\ 2\pi} \int_0^a \frac{ar^2}{\sqrt{a^2 - r^2}} rdr = 4a\pi \int_0^a \frac{r^2}{\sqrt{a^2 - r^2}} rdr$$

Metodom smene ćemo rešiti ovaj integral bez granica (lakše malo)

$$\int \frac{r^2}{\sqrt{a^2 - r^2}} rdr = \left| \begin{array}{l} a^2 - r^2 = t^2 \to r^2 = a^2 - t^2 \\ -2rdr = 2tdt \\ rdr = -tdt \end{array} \right| = \int \frac{a^2 - t^2}{t}(-t)\, dt = \int (t^2 - a^2) dt = \frac{t^3}{3} - a^2 t =$$

$$= \frac{\left(\sqrt{a^2 - r^2}\right)^3}{3} - a^2\sqrt{a^2 - r^2}$$

Sad mu ubacimo granice:

$$\left(\frac{\left(\sqrt{a^2 - r^2}\right)^3}{3} - a^2\sqrt{a^2 - r^2} \right) \Bigg|_0^a = (a^3 - \frac{a^3}{3}) - (0 - 0) = \frac{2a^3}{3}$$

Vratimo se u zadatak:

$$I = 4a\pi\int_0^a \frac{ar^2}{\sqrt{a^2-r^2}}rdr = 4a\pi\cdot\frac{2a^3}{3} = \boxed{\frac{8a^4\pi}{3}}$$ i evo konačnog rešenja!

3. **Rešiti integral** $I = \iint_S \frac{1}{x^2+y^2+z^2}dS$ **ako je S deo cilindra** $x^2+y^2=R^2$ **ograničenog ravnima x=0, y=0, z=0 i z = m.**

Rešenje:

Primećujemo da je cilindar $x^2+y^2=R^2$ uz z osu i da ne možemo odavde izraziti z. Onda ćemo izraziti ili x ili y

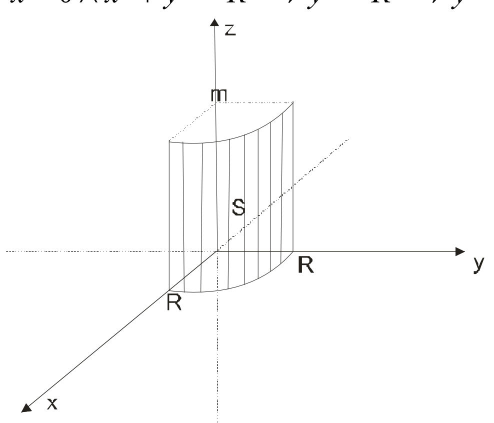
i raditi po njima sve isto kao i po z....

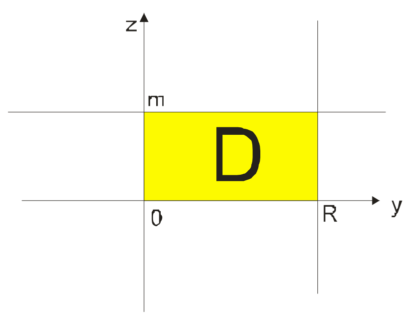
$$x^2+y^2=R^2$$
$$x^2=R^2-y^2$$
$$x=\pm\sqrt{R^2-y^2}$$ nama treba x > 0 pa je:
$$x=+\sqrt{R^2-y^2}$$
Odavde imamo :
$$p=\frac{\partial x}{\partial y}=\frac{1}{2\sqrt{R^2-y^2}}\cdot\left(R^2-y^2\right)'_{po\ y}=\frac{1}{2\sqrt{R^2-y^2}}\cdot(-2y)=\frac{-y}{\sqrt{R^2-y^2}}\cdot$$
$$q=\frac{\partial x}{\partial z}=0$$

$$\sqrt{1+p^2+q^2}=\sqrt{1+\left(\frac{-y}{\sqrt{R^2-y^2}}\right)^2+0}=\sqrt{1+\frac{y^2}{R^2-y^2}}=\sqrt{\frac{R^2-y^2+y^2}{R^2-y^2}}=\frac{R}{\sqrt{R^2-y^2}}$$

Pošto smo odabrali da radimo po x , da bi odredili granice integrala, spuštamo se u ravan x = 0 .

$$x=0\wedge x^2+y^2=R^2\rightarrow y^2=R^2\rightarrow y=\pm R\rightarrow y=R$$





$$D:\begin{cases} 0 \le y \le R \\ 0 \le z \le m \end{cases}$$

Sad da rešimo zadati integral:

$$I = \iint\limits_S \frac{1}{x^2+y^2+z^2} dS = \iint\limits_D \frac{1}{R^2 - \cancel{y^2} + \cancel{y^2} + z^2} \sqrt{1+p^2+q^2}\, dydz =$$

$$= \int_0^R \frac{Rdy}{\sqrt{R^2-y^2}} \int_0^m \frac{dz}{R^2+z^2} = \text{na stranu sredimo:}$$

$$\int_0^m \frac{dz}{R^2+z^2} = \frac{1}{R} arctg \frac{z}{R}\Bigg|_0^m = \frac{1}{R} arctg \frac{m}{R} - \frac{1}{R} arctg \frac{0}{R} = \boxed{\frac{1}{R} arctg \frac{m}{R}}$$

$$I = \frac{1}{\cancel{R}} arctg \frac{m}{R} \int_0^R \frac{\cancel{R}dy}{\sqrt{R^2-y^2}} = arctg \frac{m}{R} \cdot \int_0^R \frac{dy}{\sqrt{R^2-y^2}} = ... = \boxed{\frac{\pi}{2} arctg \frac{m}{R}}$$

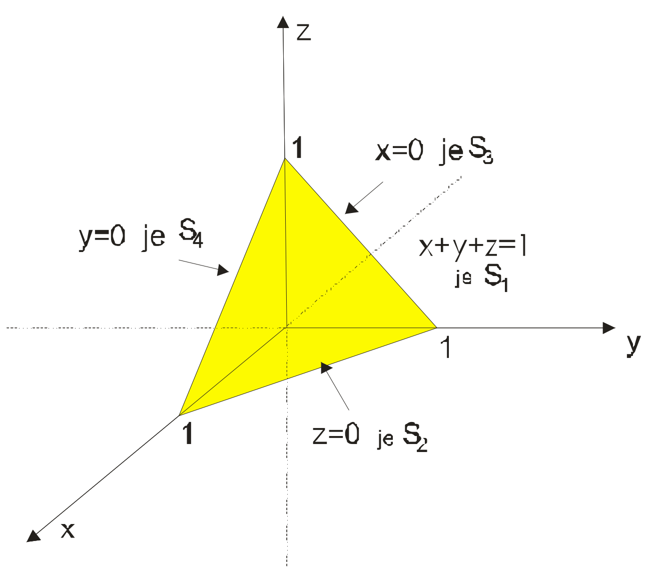
Na stranu : $\displaystyle\int_0^R \frac{dy}{\sqrt{R^2-y^2}} = \arcsin \frac{y}{R}\Bigg|_0^R = \arcsin \frac{R}{R} - \arcsin \frac{0}{R} = \arcsin 1 - \arcsin 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$$I = \frac{1}{\cancel{R}} arctg \frac{m}{R} \int_0^R \frac{\cancel{R}dy}{\sqrt{R^2-y^2}} = arctg \frac{m}{R} \cdot \int_0^R \frac{dy}{\sqrt{R^2-y^2}} = \boxed{\frac{\pi}{2} arctg \frac{m}{R}}$$

4. Rešiti integral $I = \iint\limits_S \frac{1}{(1+x+y)^2} dS$ **ako je S površ tetraedra ograničenog ravnima** $x+y+z=1, x=0, y=0, z=0$

Rešenje:

Nacrtajmo sliku i postavimo problem.



Ovde moramo raditi 4 integrala, za svaku površ posebno. **Dakle** $I = \iint\limits_{S_1} + \iint\limits_{S_2} + \iint\limits_{S_3} + \iint\limits_{S_4}$

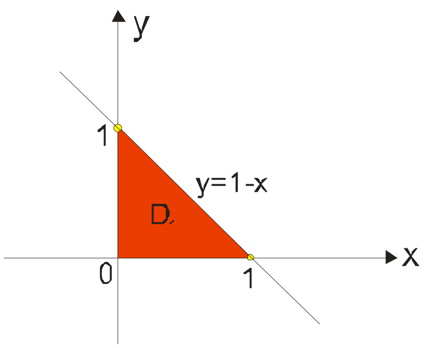
Za S_1 (ravan $x+y+z=1$) imamo:

$$x+y+z=1 \to z=1-x-y \to \frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=-1$$

$$\sqrt{1+p^2+q^2}=\sqrt{1+1+1}=\sqrt{3}$$

Ako se spustimo u ravan z = 0 imamo $x+y=1 \to y=1-x$

Nacrtajmo sliku i odredimo granice:



$$D_1:\begin{cases} 0\le x\le 1 \\ 0\le y\le 1-x \end{cases}$$

$$\iint_{S_1}\frac{1}{(1+x+y)^2}dS=\iint_{D_1}\frac{1}{(1+x+y)^2}\sqrt{3}dxdy=\sqrt{3}\int_0^1 dx\int_0^{1-x}\frac{1}{(1+x+y)^2}dy=$$

Na stranu :

$$\int\frac{1}{(1+x+y)^2}dy=\begin{vmatrix}1+x+y=t\\ dy=dt\end{vmatrix}=\int t^{-2}dt=\frac{t^{-1}}{-1}=-\frac{1}{t}=-\frac{1}{1+x+y}, \textit{pa je}$$

$$-\frac{1}{1+x+y}\Bigg|_0^{1-x}=-\frac{1}{1+x+1-x}-\left(-\frac{1}{1+x+0}\right)=-\frac{1}{2}+\frac{1}{1+x}$$

vratimo se u integral

$$\sqrt{3}\int_0^1 dx\int_0^{1-x}\frac{1}{(1+x+y)^2}dy=\sqrt{3}\int_0^1\left(-\frac{1}{2}+\frac{1}{1+x}\right)dx=\sqrt{3}\left(-\frac{1}{2}x+\ln|1+x|\right)\Bigg|_0^1=\sqrt{3}(\ln 2-\frac{1}{2})$$

Za S_2 (ravan $z=0$) imamo:

$$z=0\to\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0$$

$$\sqrt{1+p^2+q^2}=\sqrt{1}=1$$

$$x+y=1\to y=1-x$$

$$D_1:\begin{cases} 0\le x\le 1 \\ 0\le y\le 1-x \end{cases}$$

Dakle, oblast je ista kao za prethodni deo.... A i integral se slično rešava!

$$\iint_{S_2} \frac{1}{(1+x+y)^2} dS = \iint_{D_1} \frac{1}{(1+x+y)^2} dxdy = \int_0^1 dx \int_0^{1-x} \frac{1}{(1+x+y)^2} dy = \ln 2 - \frac{1}{2}$$

Za S_3 (ravan $x=0$) imamo:

$$x = 0 \to \frac{\partial x}{\partial z} = \frac{\partial x}{\partial y} = 0$$

$$\sqrt{1+p^2+q^2} = \sqrt{1} = 1$$

$$y + z = 1 \to z = 1 - y$$

$$D_3 : \begin{cases} 0 \le y \le 1 \\ 0 \le z \le 1 - y \end{cases}$$

$$\iint_{S_3} \frac{1}{(1+x+y)^2} dS = \iint_{D_3} \frac{1}{(1+0+y)^2} dydz = \int_0^1 dy \int_0^{1-y} \frac{1}{(1+y)^2} dy = 1 - \ln 2$$

Za S_4 (ravan $y=0$) imamo:

$$y = 0 \to \frac{\partial y}{\partial z} = \frac{\partial y}{\partial x} = 0$$

$$\sqrt{1+p^2+q^2} = \sqrt{1} = 1$$

$$x + z = 1 \to z = 1 - x$$

$$D_4 : \begin{cases} 0 \le x \le 1 \\ 0 \le z \le 1 - x \end{cases}$$

$$\iint_{S_4} \frac{1}{(1+x+y)^2} dS = \iint_{D_4} \frac{1}{(1+x+0)^2} dxdz = \int_0^1 dx \int_0^{1-x} \frac{1}{(1+x)^2} dz = 1 - \ln 2$$

E sad ćemo sabrati sva 4 rešenja:

$$I = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} = \sqrt{3}(\ln 2 - \frac{1}{2}) + \left(\ln 2 - \frac{1}{2}\right) + \left(1 - \ln 2\right) + \left(1 - \ln 2\right) = \boxed{\left(\sqrt{3} - 1\right)\ln 2 + \frac{3 - \sqrt{3}}{2}}$$

5. Rešiti integral $I = \iint_S z^2 dS$ **ako je S:** $\begin{cases} x = r\cos\varphi\sin\alpha \\ y = r\sin\varphi\sin\alpha \\ z = r\cos\alpha \\ 0 \le \varphi \le 2\pi \wedge 0 \le r \le a \wedge \alpha = const. \end{cases}$

Rešenje:

<u>Da se podsetimo:</u>

i) Ako je S deo po deo glatka dvostrana površ zadata jednačinama:

x=x(u,v)
y=y(u,v)
z=z(u,v)

gde (u,v) pripada D a funkcija f(x,y,z) je definisana i neprekidna na površi S , onda je:

$$\iint_S f(x,y,z)ds = \iint_D f[x(u,v),y(u,v),z(u,v)]\sqrt{EG-F^2}\,dudv$$

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

Malo ćemo "**korigovati**" formulice i upotrebiti ih u ovoj situaciji!

Najpre da nadjemo parcijalne izvode koji nam trebaju :

$$x = r\cos\varphi\sin\alpha \to \frac{\partial x}{\partial r} = \cos\varphi\sin\alpha \wedge \frac{\partial x}{\partial \varphi} = -r\sin\varphi\sin\alpha$$

$$y = r\sin\varphi\sin\alpha \to \frac{\partial y}{\partial r} = \sin\varphi\sin\alpha \wedge \frac{\partial x}{\partial \varphi} = r\cos\varphi\sin\alpha$$

$$z = r\cos\alpha \to \frac{\partial z}{\partial r} = \cos\alpha \wedge \frac{\partial z}{\partial \varphi} = 0$$

Sad tražimo E,G i F

$$E=\left(\frac{\partial x}{\partial r}\right)^2+\left(\frac{\partial y}{\partial r}\right)^2+\left(\frac{\partial z}{\partial r}\right)^2=(\cos\varphi\sin\alpha)^2+(\sin\varphi\sin\alpha)^2+(\cos\alpha)^2=$$

$$=\cos^2\varphi\sin^2\alpha+\sin^2\varphi\sin^2\alpha+\cos^2\alpha=\sin^2\alpha\Big(\underbrace{\cos^2\varphi+\sin^2\varphi}_{ovo\ je\ 1}\Big)+\cos^2\alpha=1$$

$$G=\left(\frac{\partial x}{\partial \varphi}\right)^2+\left(\frac{\partial y}{\partial \varphi}\right)^2+\left(\frac{\partial z}{\partial \varphi}\right)^2=(-r\sin\varphi\sin\alpha)^2+(r\cos\varphi\sin\alpha)^2+0=$$

$$=r^2\sin^2\varphi\sin^2\alpha+r^2\cos^2\varphi\sin^2\alpha=r^2\sin^2\alpha\left(\sin^2\varphi+\cos^2\varphi\right)=r^2\sin^2\alpha$$

$$F=\frac{\partial x}{\partial r}\frac{\partial x}{\partial \varphi}+\frac{\partial y}{\partial r}\frac{\partial y}{\partial \varphi}+\frac{\partial z}{\partial r}\frac{\partial z}{\partial \varphi}=$$

$$=\cos\varphi\sin\alpha\cdot(-r\sin\varphi\sin\alpha)+\sin\varphi\sin\alpha\cdot r\cos\varphi\sin\alpha+0=0$$

U zadatku nam je već dato da je: $D:\begin{cases}0\le\varphi\le 2\pi\\ 0\le r\le a\end{cases}$

Sad rešavamo po formuli:

$$I=\iint_S z^2 dS=\iint_D (r\cos\alpha)^2\sqrt{EG-F^2}\,drd\varphi=\int_0^{2\pi}d\varphi\int_0^a r^2\cos^2\alpha\sqrt{r^2\sin^2\alpha}\,dr=\int_0^{2\pi}d\varphi\int_0^a r^2\cos^2\alpha\cdot r|\sin\alpha|\,dr=$$

= pazite, i α je konstanta , pa sve ide ispred integrala!

$$=\cos^2\alpha|\sin\alpha|\int_0^{2\pi}d\varphi\int_0^a r^3 dr=\cos^2\alpha|\sin\alpha|\cdot 2\pi\cdot\frac{a^4}{4}=\boxed{\cos^2\alpha|\sin\alpha|\cdot\frac{a^4\pi}{2}}$$

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