VIŠESTRUKI INTEGRALI - ZADACI (II DEO)

Dvostruki integrali-rešavanje

Primer 1.

Izračunati integral $\iint\limits_D \frac{x^2}{1+y^2}dxdy$ ako je $0 \le x \le 1$ i $0 \le y \le 1$

Rešenje:

Ovde su nam odmah date granice integrala pa ne moramo crtati sliku i odredjivati ih.

Uvek je pitanje da li je lakše raditi prvo po x pa po y ili obrnuto...

Pogledajte najpre dati dvojni integral, razmislite malo pa tek onda krenite sa radom...

Mi ćemo prvo rešavati ovaj integral po x pa onda po y...

$\iint\limits_D \frac{x^2}{1+y^2}dxdy = \int\limits_0^1 dy \int\limits_0^1 \frac{x^2}{1+y^2}dx$ neki profesori vole da zapišu i ovako:

$\iint\limits_D \frac{x^2}{1+y^2}dxdy = \int\limits_0^1 \left(\int\limits_0^1 \frac{x^2}{1+y^2}dx \right) dy$ Vi naravno radite kao što zahteva vaš profesor...

Dakle, najpre rešavamo integral u zagradi. On je "po x" pa ovde ***y* tretiramo kao konstantu**!

$$\iint\limits_D \frac{x^2}{1+y^2}dxdy = \int\limits_0^1 \left(\int\limits_0^1 \frac{x^2}{1+y^2}dx \right) dy = \int\limits_0^1 \frac{1}{1+y^2} \left(\int\limits_0^1 x^2 dx \right) dy =$$

Rešićemo ga " na stranu " ...

$$\int\limits_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

Sad ovo rešenje ubacimo u dvojni integral:

$$\iint\limits_D \frac{x^2}{1+y^2}dxdy = \int\limits_0^1 \left(\int\limits_0^1 \frac{x^2}{1+y^2}dx \right) dy = \int\limits_0^1 \frac{1}{1+y^2} \left(\int\limits_0^1 x^2 dx \right) dy = \int\limits_0^1 \frac{1}{1+y^2}dy =$$

$$= \frac{1}{3}\int\limits_0^1 \frac{1}{1+y^2}dy = \frac{1}{3} arctgy \Big|_0^1 = \frac{1}{3}\left(arctg1 - arctg0 \right) = \frac{1}{3}(\frac{\pi}{4} - 0) = \boxed{\frac{\pi}{12}}$$

I evo rešenja.

Primer 2.

Izračunati integral $\iint\limits_D x\sin(x+y)dxdy$ ako je $0\le x\le \pi$ i $0\le y\le \frac{\pi}{2}$

Rešenje:

I ovde odmah imamo granice, pa slika ne treba...

$$\iint\limits_D x\sin(x+y)dxdy=\int_0^{\pi} xdx\int_0^{\frac{\pi}{2}}\sin(x+y)dy=\int_0^{\pi} x\left(\int_0^{\frac{\pi}{2}}\sin(x+y)dy\right)dx$$

Integral u zagradi rešimo na stranu:

$$\int_0^{\frac{\pi}{2}}\sin(x+y)dy=-\cos(x+y)\Big|_0^{\frac{\pi}{2}}=-[\cos(x+\frac{\pi}{2})-\cos(x+0)]=-[\cos(x+\frac{\pi}{2})-\cos x]$$

Iz trigonometrije znamo da je $\cos(x+\frac{\pi}{2})=-\sin x$ pa je

$$\int_0^{\frac{\pi}{2}}\sin(x+y)dy=-\cos(x+y)\Big|_0^{\frac{\pi}{2}}=-[\cos(x+\frac{\pi}{2})-\cos(x+0)]=-[\cos(x+\frac{\pi}{2})-\cos x]=-[-\sin x-\cos x]$$
$$=\sin x+\cos x$$

Vratimo se u dvojni integral:

$$\iint\limits_D x\sin(x+y)dxdy=\int_0^{\pi} xdx\int_0^{\frac{\pi}{2}}\sin(x+y)dy=\int_0^{\pi} x\left(\int_0^{\frac{\pi}{2}}\sin(x+y)dy\right)dx=\int_0^{\pi} x(\sin x+\cos x)dx$$

Ovde imamo parcijalnu integraciju , za oba integrala...

$$\int_0^{\pi} x\sin xdx=\begin{vmatrix} x=u & \sin xdx=dv \\ dx=du & -\cos x=v\end{vmatrix}=-x\cos x-\int(-\cos x)dx=$$
$$=\left(-x\cos x+\sin x\right)\Big|_0^{\pi}=\left(-\pi\cos\pi+\sin\pi\right)-\left(-0\cos 0+\sin 0\right)=-\pi(-1)=\pi$$

$$\int_0^{\pi} x\cos x dx = \begin{vmatrix} x=u & \cos x dx = dv \\ dx = du & \sin x = v \end{vmatrix} = x\sin x - \int \sin dx =$$

$$= \left(x\sin x + \cos x\right)\Big|_0^{\pi} = \left(\pi\sin\pi + \cos\pi\right) - \left(0\sin 0 + \cos 0\right) = -1 - 1 = -2$$

Rešenje će biti:

$$\iint_D x\sin(x+y)dxdy = \int_0^{\pi} xdx \int_0^{\frac{\pi}{2}} \sin(x+y)dy = \int_0^{\pi} x\left(\int_0^{\frac{\pi}{2}} \sin(x+y)dy\right)dx = \int_0^{\pi} x(\sin x + \cos x)dx = \boxed{\pi - 2}$$

Kao što ste videli ovaj dvojni integral smo prvo rešavali po y pa onda po x.

Da li bi bilo lakše da smo išli obrnuto?

Da vidimo:

II način

$$\iint_D x\sin(x+y)dxdy = \int_0^{\frac{\pi}{2}}\left(\int_0^{\pi} x\sin(x+y)dx\right)dy$$

Integral u zagradi rešićemo na stranu , kao neodredjeni , pa ćemo mu dodati granice:

$$\int_0^{\pi} x\sin(x+y)dx = ?$$

$$\int x\sin(x+y)dx = \begin{vmatrix} x=u & \sin(x+y)dx = dv \\ dx = du & -\cos(x+y) = v \end{vmatrix} = -x\cos(x+y) - \int[-\cos(x+y)]dx =$$

$$= -x\cos(x+y) + \sin(x+y)$$

$$\int_0^{\pi} x\sin(x+y)dx = -x\cos(x+y) + \sin(x+y)\Big|_0^{\pi} = [-\pi\cos(\pi+y) + \sin(\pi+y)] - [-0\cos(0+y) + \sin(0+y)] =$$

$$= -\pi\cos(\pi+y) + \sin(\pi+y) - \sin y$$

$$\iint_D x\sin(x+y)dxdy = \int_0^{\frac{\pi}{2}}\left(\int_0^{\pi} x\sin(x+y)dx\right)dy = \int_0^{\frac{\pi}{2}}\left(-\pi\cos(\pi+y) + \sin(\pi+y) - \sin y\right)dy =$$

$$= \left(-\pi\sin(\pi+y) - \cos(\pi+y) + \cos y\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= \left(-\pi\sin(\pi+\frac{\pi}{2}) - \cos(\pi+\frac{\pi}{2}) + \cos\frac{\pi}{2}\right) - \left(-\pi\sin(\pi+0) - \cos(\pi+0) + \cos 0\right) =$$

$$= \pi - 0 + 0 - (0 + 1 + 1) = \boxed{\pi - 2}$$

Možda malo brže...Bitno je da je rešenje dobro!

Primer 3.

Izračunati integral $\iint_D xy^2dxdy$ ako je oblast integracije ograničena parabolom $y^2=2x$ i pravom $x=\frac{1}{2}$

Rešenje:

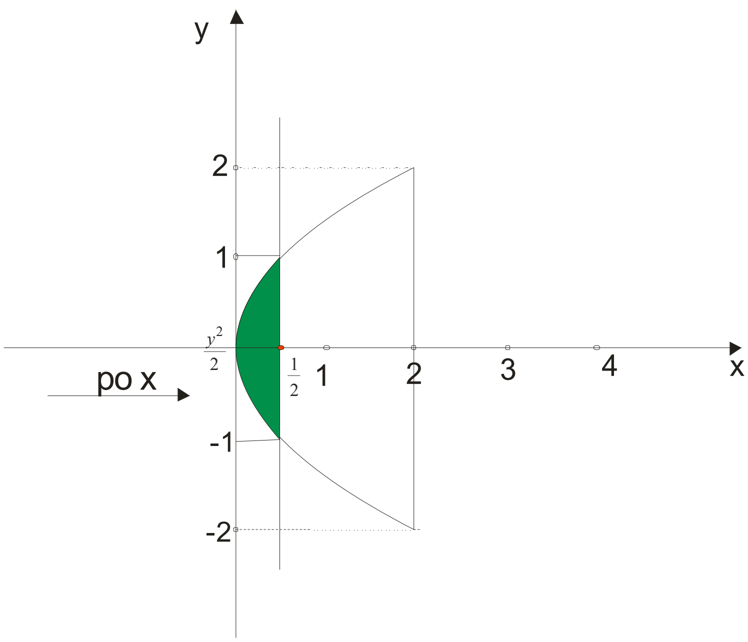
Najpre ćemo odrediti preseke i nacrtati sliku:

Presek odredjujemo rešavajući sistem jednačina.

$$y^2=2x$$

$$x=\frac{1}{2}$$

$$y^2=2\cdot\frac{1}{2}\to y^2=1\to y=\pm1\to M(\frac{1}{2},-1)\wedge N(\frac{1}{2},1)$$



Slika nam pomaže da odredimo odlast $D:\begin{cases}-1\le y\le1\\ \frac{y^2}{2}\le x\le\frac{1}{2}\end{cases}$ (pogledajte prethodni fajl)

Sad imamo:

$$\iint_D xy^2dxdy=\int_{-1}^{1}dy\int_{\frac{y^2}{2}}^{\frac{1}{2}}xy^2dx=\int_{-1}^{1}\left(\int_{\frac{y^2}{2}}^{\frac{1}{2}}xy^2dx\right)dy$$ Prvo rešavamo integral u zagradi:

$$\int_{\frac{y^2}{2}}^{\frac{1}{2}} xy^2 dx = y^2 \cdot \int_{\frac{y^2}{2}}^{\frac{1}{2}} x dx = y^2 \cdot [\frac{x^2}{2}]\Bigg|_{\frac{y^2}{2}}^{\frac{1}{2}} = y^2 \cdot [\frac{\left(\frac{1}{2}\right)^2}{2} - \frac{\left(\frac{y^2}{2}\right)^2}{2}] = \frac{y^2}{2}[\frac{1}{4} - \frac{y^4}{4}] = \frac{1}{8}[y^2 - y^6]$$

Vraćamo se u dvojni integral:

$$\iint_D xy^2 dxdy = \int_{-1}^{1} dy \int_{\frac{y^2}{2}}^{\frac{1}{2}} xy^2 dx = \int_{-1}^{1} \left(\frac{1}{8}[y^2 - y^6]\right) dy =$$

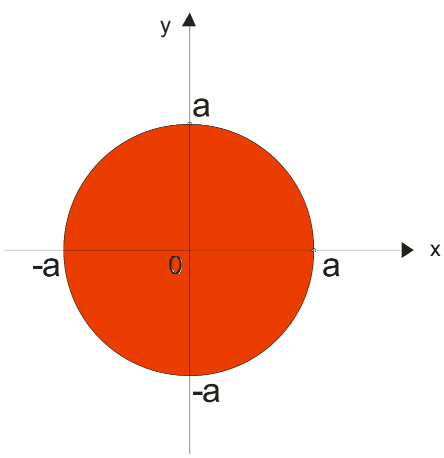
$$= \frac{1}{8}[\frac{y^3}{3} - \frac{y^7}{7}]\Bigg|_{-1}^{1} = \frac{1}{8}\{[\frac{1^3}{3} - \frac{1^7}{7}] - [\frac{(-1)^3}{3} - \frac{(-1)^7}{7}]\} = \frac{1}{8} \cdot \frac{8}{21} = \boxed{\frac{1}{21}}$$

Primer 4.

Izračunati $\iint_D \sqrt{x^2 + y^2}\, dxdy$ ako je oblast D zadata sa $x^2 + y^2 \le a^2$

Rešenje:

Pogledajmo sliku:



U ovakvim slučajevima, kad je zadata kružnica, zgodno je preći na polarne koordinate:

$x = r\cos\varphi$

$y = r\sin\varphi$

$|J| = r$

I onda se integral rešava:

$$\iint_D z(x,y)dxdy = \iint_{D`} z(r\cos\varphi, r\sin\varphi)|J| drd\varphi = \int_{\varphi_1}^{\varphi_2} d\varphi \int_0^r z(r\cos\varphi, r\sin\varphi) rdr$$

Najpre da odredimo granice:

$$x^2+y^2=a^2$$
$$(r\cos\varphi)^2+(r\sin\varphi)^2=a^2$$
$$r^2(\cos^2\varphi+\sin^2\varphi)=a^2 \quad \text{znamo da je } \cos^2\varphi+\sin^2\varphi=1$$
$$r^2=a^2 \to r=a$$

Dakle r ide od 0 da a.

Pošto nam ovde treba ceo krug, jasno je da $0\le\varphi\le 2\pi$

Imamo dakle da je D`= $\begin{cases} 0\le r\le a \\ 0\le\varphi\le 2\pi \end{cases}$ pa je :

$$\iint_D \sqrt{x^2+y^2}\,dxdy=\int_0^{2\pi}d\varphi\int_0^a\sqrt{r^2}\,rdr=\int_0^{2\pi}\left(\int_0^a\sqrt{r^2}\,rdr\right)d\varphi=\int_0^{2\pi}\left(\int_0^a r^2dr\right)d\varphi=$$

Kao i uvek , integral u zagradi rešimo posebno...

$$\int_0^a r^2dr=\frac{r^3}{3}\Big|_0^a=\frac{a^3}{3}$$

Sad imamo:

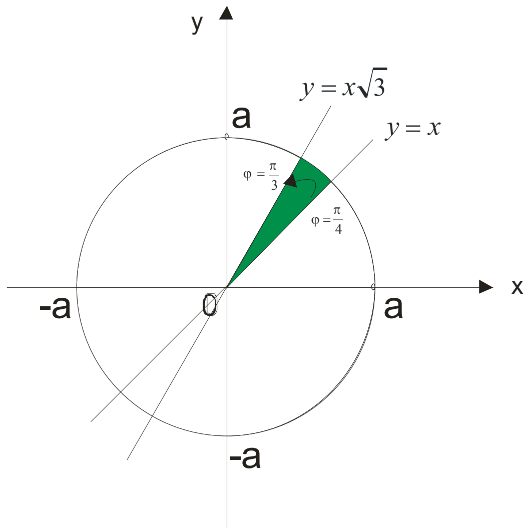
$$\iint_D \sqrt{x^2+y^2}\,dxdy=\int_0^{2\pi}d\varphi\int_0^a\sqrt{r^2}\,rdr=\int_0^{2\pi}\left(\int_0^a\sqrt{r^2}\,rdr\right)d\varphi=\int_0^{2\pi}\left(\int_0^a r^2dr\right)d\varphi=$$
$$=\int_0^{2\pi}\frac{a^3}{3}d\varphi=\frac{a^3}{3}\int_0^{2\pi}d\varphi=\frac{a^3}{3}\cdot\varphi\Big|_0^{2\pi}=\frac{a^3}{3}(2\pi-0)=\boxed{\frac{2\pi a^3}{3}}$$

Primer 5.

Izračunati $\iint_D\sqrt{a^2-x^2-y^2}\,dxdy$ ako je oblast D ograničena sa $x^2+y^2=a^2$, $y=x, y=x\sqrt{3}$ u prvom kvadrantu.

<u>Rešenje:</u>

Nacrtajmo sliku:



Koristićemo polarne koordinate:

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$|J| = r$$

Odredimo granice za r i φ:

$$x^2 + y^2 = a^2$$
$$(r\cos\varphi)^2 + (r\sin\varphi)^2 = a^2$$
$$r^2(\cos^2\varphi + \sin^2\varphi) = a^2$$
$$r^2 = a^2 \rightarrow r = a$$

Dakle: r ide od 0 da a.

Iz pravih $y = x, y = x\sqrt{3}$ ćemo odrediti odakle dokle ide ugao φ

Da je podsetimo :

Prava $y = kx + n$ ima koeficijent pravca $k = tg\varphi$.

Iz prave $y = x$ je $k=1$ pa je $tg\varphi = 1 \rightarrow \varphi = \frac{\pi}{4}$

Iz prave $y = x\sqrt{3}$ je $k = \sqrt{3}$ pa je $tg\varphi = \sqrt{3} \rightarrow \varphi = \frac{\pi}{3}$

Dobili smo dakle da je :

$$D` = \begin{cases} 0 \le r \le a \\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{3} \end{cases}$$

Da rešimo sada integral:

$$\iint_D \sqrt{a^2-x^2-y^2}\,dxdy = \iint_D \sqrt{a^2-(x^2+y^2)}\,dxdy = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^a \sqrt{a^2-r^2}\cdot rdr$$

$\int_0^a \sqrt{a^2-r^2}\cdot rdr$ ćemo rešiti na stranu i najpre bez granica...

$$\int \sqrt{a^2-r^2}\cdot rdr = \begin{vmatrix} a^2-r^2=t^2 \\ -2rdr=2tdt \\ rdr=-tdt \end{vmatrix} = \int \sqrt{t^2}(-t)dt = -\int t^2 dt = -\frac{t^3}{3} = -\frac{\left(\sqrt{a^2-r^2}\right)^3}{3}$$

$$\int_0^a \sqrt{a^2-r^2}\cdot rdr = -\frac{\left(\sqrt{a^2-r^2}\right)^3}{3}\Bigg|_0^a = -[\frac{\left(\sqrt{a^2-a^2}\right)^3}{3} - \frac{\left(\sqrt{a^2-0^2}\right)^3}{3}] = -[-\frac{a^3}{3}] = \frac{a^3}{3}$$

$$\iint_D \sqrt{a^2-x^2-y^2}\,dxdy = \iint_D \sqrt{a^2-(x^2+y^2)}\,dxdy = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^a \sqrt{a^2-r^2}\cdot rdr =$$

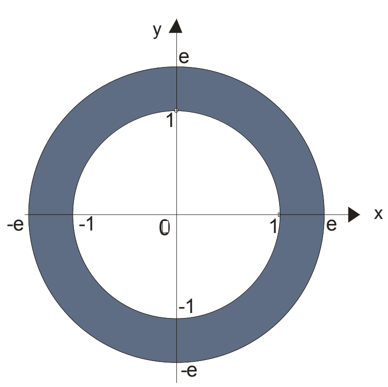
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{a^3}{3} d\varphi = \frac{a^3}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi = \frac{a^3}{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{a^3}{3}\frac{\pi}{12} = \boxed{\frac{a^3\pi}{36}}$$

Primer 6.

Izračunati $\iint_D \frac{\ln(x^2+y^2)}{x^2+y^2}dxdy$ ako je oblast D izmedju krugova $x^2+y^2=1$ i $x^2+y^2=e^2$

Rešenje:

Slika:



Uzimamo polarne koordinate:

$$x = r\cos\varphi \qquad x^2+y^2=1 \qquad x^2+y^2=e^2$$
$$y = r\sin\varphi \qquad (r\cos\varphi)^2+(r\sin\varphi)^2=1 \qquad (r\cos\varphi)^2+(r\sin\varphi)^2=e^2$$
$$|J| = r \qquad r^2(\cos^2\varphi+\sin^2\varphi)=1 \qquad r^2(\cos^2\varphi+\sin^2\varphi)=e^2$$
$$r^2=1 \to r=1 \qquad r^2=e^2 \to r=e$$

Dakle , imamo da $1 \le r \le e$

Sa slike vidimo da ugao uzima pun krug $0 \le \varphi \le 2\pi$.

$$\iint_D \frac{\ln(x^2+y^2)}{x^2+y^2}dxdy = \int_0^{2\pi} d\varphi \int_1^e \frac{\ln r^2}{r^{\not 2}}\cdot \not r\, dr = \int_0^{2\pi} d\varphi \int_1^e \frac{2\ln r}{r}dr$$

$$\int_1^e \frac{2\ln r}{r}dr = 2\int_1^e \frac{\ln r}{r}dr$$

$$\int \frac{\ln r}{r}dr = \begin{vmatrix} \ln r = t \\ \frac{1}{r}dr = dt \end{vmatrix} = \int t\,dt = \frac{t^2}{2} = \frac{\ln^2 r}{2}$$

$$\int_1^e \frac{2\ln r}{r}dr = 2\int_1^e \frac{\ln r}{r}dr = \not 2\, \frac{\ln^2 r}{\not 2}\Bigg|_1^e = \ln^2 e - \ln^2 1 = 1-0 = 1$$

$$\iint_D \frac{\ln(x^2+y^2)}{x^2+y^2}dxdy = \int_0^{2\pi} d\varphi \int_1^e \frac{\ln r^2}{r^{\not 2}}\cdot \not r\, dr = \int_0^{2\pi} d\varphi \int_1^e \frac{2\ln r}{r}dr = \int_0^{2\pi} d\varphi = 2\pi - 0 = \boxed{2\pi}$$

Primer 7.

Izračunati integral $\iint_D xydxdy$, gde je oblast D ograničena Ox osom i lukovima krugova:

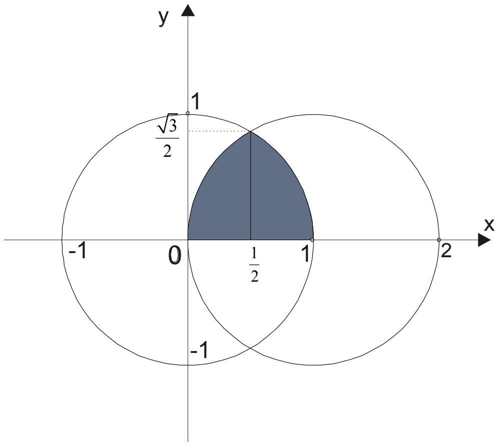
$x^2+y^2=1$ i $x^2+y^2-2x=0$ u prvom kvadrantu.

Rešenje:

Da spakujemo drugu kružnicu, nadjemo preseke i nacrtamo sliku:

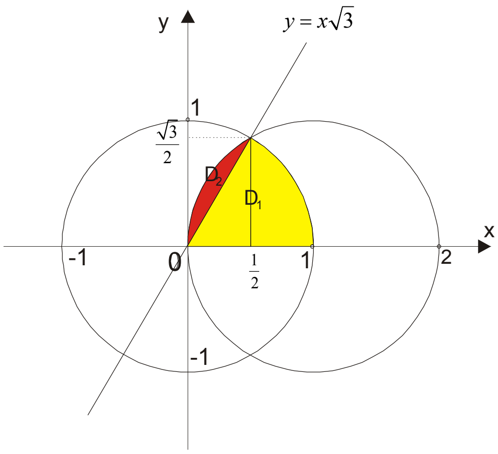
$$x^2+y^2-2x=0$$
$$x^2-2x+1-1+y^2=0$$
$$(x-1)^2+y^2=1$$

Presek je: $x^2+y^2-2x=0 \wedge x^2+y^2=1 \to 1-2x=0 \to x=\frac{1}{2} \to y=\frac{\sqrt{3}}{2}$



Ovu oblast moramo podeliti na dva dela:

Prava koja prolazi kroz tačku preseka krugova i koordinatni početak je $y = \sqrt{3}x$ (kao jednačina prave kroz dve tačke, pogledajte prethodni fajl)



Opet ćemo preći na polarne koordinate:

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$|J| = r$$

Za oblast D_1 imamo :

$$x^2 + y^2 = 1$$
$$(r\cos\varphi)^2 + (r\sin\varphi)^2 = 1$$
$$r^2(\cos^2\varphi + \sin^2\varphi) = 1$$
$$r^2 = 1 \to r = 1$$

Dakle: $0 \le r \le 1$

Ugao ide od x ose do prave $y = \sqrt{3}x$ pa je $0 \le \varphi \le \frac{\pi}{3}$

Za oblast D_2 imamo :

$$x^2 + y^2 - 2x = 0$$
$$(r\cos\varphi)^2 + (r\sin\varphi)^2 - 2r\cos\varphi = 0$$
$$r^2(\cos^2\varphi + \sin^2\varphi) = 2r\cos\varphi$$
$$r^2 = 2r\cos\varphi \to r = 2\cos\varphi$$

Odavde zaključujemo: $0 \le r \le 2\cos\varphi$

Ugao ide od prave $y = \sqrt{3}x$ pa do y ose pa je $\frac{\pi}{2} \le \varphi \le \frac{\pi}{3}$

Imamo dakle:

$$D_1 : \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \frac{\pi}{3} \end{cases} \qquad D_2 : \begin{cases} 0 \le r \le 2\cos\varphi \\ \frac{\pi}{2} \le \varphi \le \frac{\pi}{3} \end{cases}$$

Podintegralna funkcija će kad stavimo smene biti:

$$xy = r\cos\varphi \cdot r\sin\varphi = r^2 \sin\varphi\cos\varphi$$

Da rešavamo sada integral:

$$\iint_D xydxdy = \int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^2 \sin\varphi\cos\varphi \cdot rdr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 \sin\varphi\cos\varphi \cdot rdr =$$
$$= \int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^3 \sin\varphi\cos\varphi dr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^3 \sin\varphi\cos\varphi dr$$

Svaki ćemo posebno , pa ćemo sabrati rešenja:

$$\int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^3 \sin\varphi\cos\varphi dr = \int_0^{\frac{\pi}{3}} \sin\varphi\cos\varphi d\varphi \int_0^1 r^3 dr = \int_0^{\frac{\pi}{3}} s\,in\varphi\cos\varphi d\varphi \cdot \frac{r^4}{4}\Big|_0^1 = \frac{1}{4}\int_0^{\frac{\pi}{3}} s\,in\varphi\cos\varphi d\varphi$$

$$\int s\,in\varphi\cos\varphi d\varphi = \begin{vmatrix} \sin\varphi = t \\ \cos\varphi d\varphi = dt \end{vmatrix} = \int tdt = \frac{t^2}{2} = \frac{\sin^2\varphi}{2}$$

$$\int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^3 \sin\varphi\cos\varphi dr = \int_0^{\frac{\pi}{3}} \sin\varphi\cos\varphi d\varphi \int_0^1 r^3 dr = \int_0^{\frac{\pi}{3}} s\,in\varphi\cos\varphi d\varphi \cdot \frac{r^4}{4}\Big|_0^1 = \frac{1}{4}\frac{\sin^2\varphi}{2}\Big|_0^{\frac{\pi}{3}} =$$

$$= \frac{1}{8}\left(\sin^2\frac{\pi}{3} - \sin^2 0\right) = \frac{1}{8}\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{8}\left(\frac{3}{4}\right) = \boxed{\frac{3}{32}}$$

Sad drugi:

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^3 \sin\varphi\cos\varphi dr = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\cos\varphi d\varphi \int_0^{2\cos\varphi} r^3 dr = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\cos\varphi d\varphi \cdot \frac{r^4}{4}\Big|_0^{2\cos\varphi} =$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\cos\varphi \cdot \frac{16\cos^4\varphi}{4} d\varphi = 4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\cos^5\varphi d\varphi =$$

Na stranu kao neodredjeni pa vratimo granice...

$$\int \sin\varphi\cos^5\varphi d\varphi = \begin{vmatrix} \cos\varphi = t \\ -\sin\varphi d\varphi = dt \end{vmatrix} = -\int t^5 dt = -\frac{t^6}{6} = -\frac{\cos^6\varphi}{6}$$

$$4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\cos^5\varphi d\varphi = 4\left(-\frac{\cos^6\varphi}{6}\right)\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = -4[\frac{\cos^6\frac{\pi}{2}}{6} - \frac{\cos^6\frac{\pi}{3}}{6}] = -4[0 - \frac{\frac{1}{64}}{6}] = \boxed{\frac{1}{96}}$$

Konačno je:

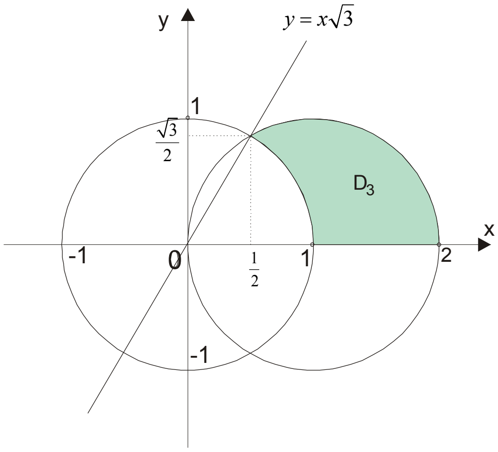
$$\iint_D xydxdy = \int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^2 \sin\varphi\cos\varphi \cdot rdr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 \sin\varphi\cos\varphi \cdot rdr = \frac{3}{32} + \frac{1}{96} = \boxed{\frac{5}{48}}$$

I ovo bi bilo rešenje našeg zadatka. **ALI!**

PREVIDELI SMO JEDNU STVAR!

Ovde postoji i druga moguća oblast!

Pogledajmo sliku opet.



I ova oblast je ograničena datim kružnicama i x osom u prvom kvadrantu!

Pazite na ovo, zadatak može biti iz dva dela a da vam to profesor ne napomene...

Ovde bi bilo:

$$D_3 : \begin{cases} 1 \le r \le 2\cos\varphi \\ 0 \le \varphi \le \dfrac{\pi}{3} \end{cases}$$

Sličnim rešavanjem kao malopre bi dobili:

$$\iint_D xydxdy = \int_0^{\frac{\pi}{3}} d\varphi \int_1^{2\cos\varphi} r^2 \sin\varphi\cos\varphi \cdot rdr = \int_0^{\frac{\pi}{3}} d\varphi \int_1^{2\cos\varphi} r^3 \sin\varphi\cos\varphi dr = \frac{9}{16}$$

Primer 8.

Izračunati vrednost integrala $\iint_D (y-x)dxdy$ ako je oblast ograničena pravama

$$y = x+1$$
$$y = x-3$$
$$y = -\frac{1}{3}x + \frac{7}{3}$$
$$y = -\frac{1}{3}x + 5$$

stavljajući da je: $\begin{aligned} u &= y - x \\ v &= y + \frac{1}{3}x \end{aligned}$

Rešenje:

Da se podsetimo:

Ako se sa x=x(u,v) i y=y(u,v) , gde su ovo neprekidne i diferencijabilne funkcije , realizuje jednoznačno preslikavanje ograničene i zatvorene oblasti D u ravni xOy na oblast D` u ravni uOv i ako je:

$$J=\frac{D(x,y)}{D(u,v)}\neq 0$$

Onda važi formula:

$$\iint_D z(x,y)dxdy=\iint_{D`} z[x(u,v),y(u,v)]|J|dudv$$

Nemamo mnogo da mozgamo, jer su nam dati *u* i *v*.

Ovde nam je prvi poso da izrazimo x i y :

$$\begin{aligned}
&u=y-x\\
&\underline{v=y+\frac{1}{3}x}..................../*3\\
&u=y-x\\
&\underline{3v=3y+x}\\
&u+3v=4y\rightarrow \boxed{y=\frac{1}{4}u+\frac{3}{4}v}\\
&u=y-x......................*(-3)\\
&\underline{3v=3y+x}\\
&-3u=-3y+3x\\
&\underline{3v=3y+x}\\
&-3u+3v=4x\rightarrow \boxed{x=-\frac{3}{4}u+\frac{3}{4}v}
\end{aligned}$$

Sad tražimo Jakobijan koji mora biti različit od nule:

$$\frac{D(x,y)}{D(u,v)}=\begin{vmatrix}-\frac{3}{4} & \frac{3}{4}\\ \frac{1}{4} & \frac{3}{4}\end{vmatrix}=-\frac{9}{16}-\frac{3}{16}=-\frac{12}{16}=-\frac{3}{4}$$

Naravno, mi uzimamo apsolutnu vrednost (po formuli $\iint_D z(x,y)dxdy=\iint_{D`} z[x(u,v),y(u,v)]|J|dudv$) to jest , kod nas je $|J|=\frac{3}{4}$

Kako odrediti granice za u i v ?

Posmatrajmo granice po x i y:

$$\left.\begin{array}{l} y = x+1 \\ y = x-3 \end{array}\right\} \rightarrow \left.\begin{array}{l} y-x=1 \\ y-x=-3 \end{array}\right\} \rightarrow \left.\begin{array}{l} u=1 \\ u=-3 \end{array}\right\} \rightarrow \boxed{-3 \le u \le 1}$$

$$\left.\begin{array}{l} y=-\frac{1}{3}x+\frac{7}{3} \\ y=-\frac{1}{3}x+5 \end{array}\right\} \rightarrow \left.\begin{array}{l} y+\frac{1}{3}x=\frac{7}{3} \\ y+\frac{1}{3}x=5 \end{array}\right\} \rightarrow \left.\begin{array}{l} v=\frac{7}{3} \\ v=5 \end{array}\right\} \rightarrow \boxed{\frac{7}{3} \le v \le 5}$$

U datom integralu moramo zameniti x i y iz onog što smo izrazili:

$$\iint_D (y-x)dxdy = \iint_{D`}\left( (\frac{1}{4}u+\frac{3}{4}v)-(-\frac{3}{4}u+\frac{3}{4}v)\right)\cdot\frac{3}{4}dudv = \iint_{D`}\left(\frac{1}{4}u+\cancel{\frac{3}{4}v}+\frac{3}{4}u-\cancel{\frac{3}{4}v}\right)\cdot\frac{3}{4}dudv =$$

$$=\iint_{D`} u\cdot\frac{3}{4}dudv = \int_{\frac{7}{3}}^{5}\left(\frac{3}{4}\int_{-3}^{1}udu\right)dv = \frac{3}{4}\int_{\frac{7}{3}}^{5}\left(\frac{u^2}{2}\Big|_{-3}^{1}\right)dv = \frac{3}{4}\int_{\frac{7}{3}}^{5}\left(\frac{1^2}{2}-\frac{(-3)^2}{2}\right)dv = \frac{3}{4}\int_{\frac{7}{3}}^{5}(-4)dv =$$

$$=-3\int_{\frac{7}{3}}^{5} dv = -3v\Big|_{\frac{7}{3}}^{5} = -3(5-\frac{7}{3}) = -3\cdot\frac{8}{3} = \boxed{-8}$$