EKSTREMUMI FUNKCIJA VIŠE PROMENLJIVIH (I deo)

Kod ovakvih zadataka najčešće se zadaje funkcija $f(x, y)$.

Prvi posao nam je da nađemo parcijalne izvode $\frac{\partial f}{\partial x}$ i $\frac{\partial f}{\partial y}$.

Zatim rešavamo sistem jednačina $\frac{\partial f}{\partial x} = 0$ i $\frac{\partial f}{\partial y} = 0$.

Rešenja ovog sistema (može da bude jedno, ali i više njih) nam daju **stacionarne tačke** (x_0, y_0), (x_1, y_1), ...itd.

Dalje tražimo: $A = \frac{\partial^2 f}{\partial x^2}, \quad B = \frac{\partial^2 f}{\partial x \partial y}, \quad C = \frac{\partial^2 f}{\partial y^2}$

Oformimo $D = A \cdot C - B^2$

Za svaku stacionarnu tačku radimo isto:

Najpre vrednosti stacionarne tačke zamenimo u D. Dobijena vrednost mora da je veća od nule $\boxed{D > 0}$, ako se desi da nije , onda ta tačka nije tačka ekstremuma.

Dalje ispitujemo da li je maksimum ili minimum:

i) Ako je $\boxed{D > 0}$ i $A < 0$ $(C < 0)$ onda je stacionarna tačka **maksimum**

ii) Ako je $\boxed{D > 0}$ i $A > 0$ $(C > 0)$ onda je stacionarna tačka **minimum**

iii) Ako je $D = 0$ slučaj je **neodređen**

iv) Ako je $D < 0$ **nema ekstremuma**

Ako se desi da je $D = 0$, to jest da je slučaj neodređen, onda moramo ići na širu definiciju ekstremuma, to jest tražimo diferencijal drugog reda $d^2 f$ i

a) Ako je $d^2 f > 0$ funkcija ima minimum

b) Ako je $d^2 f < 0$ funkcija ima maksimum

primer 1.

Naći ekstremume funkcije $z = x^3 + 8y^3 - 6xy + 5$

Rešenje:

Najpre tražimo prve parcijalne izvode:

$$z = x^3 + 8y^3 - 6xy + 5$$
$$\frac{\partial z}{\partial x} = 3x^2 - 6y$$
$$\frac{\partial z}{\partial y} = 24y^2 - 6x$$

Oformimo sistem jednačina $\frac{\partial z}{\partial x} = 0$ i $\frac{\partial z}{\partial y} = 0$.

$$3x^2 - 6y = 0 ...../:3$$
$$\underline{24y^2 - 6x = 0}....../:6$$
$$x^2 - 2y = 0 \to y = \frac{x^2}{2} \quad \text{zamenimo u} \quad 4y^2 - x = 0$$
$$\underline{4y^2 - x = 0}$$
$$4\left(\frac{x^2}{2}\right)^2 - x = 0 \to 4\frac{x^4}{4} - x = 0 \to x^4 - x = 0 \to x(x^3 - 1) = 0$$
$$x = 0 \lor x = 1$$
$$Za \quad x = 0 \to y = \frac{x^2}{2} \to y = 0 \to \boxed{M_1(0,0)}$$
$$Za \quad x = 1 \to y = \frac{x^2}{2} \to y = \frac{1}{2} \to \boxed{M_1(1,\frac{1}{2})}$$

Dobili smo dve stacionarne tačke : $M_1(0,0)$ i $M_1(1,\frac{1}{2})$.

Dalje tražimo $A = \frac{\partial^2 z}{\partial x^2}$, $B = \frac{\partial^2 z}{\partial x \partial y}$, $C = \frac{\partial^2 z}{\partial y^2}$ i pravimo $D = A \cdot C - B^2$

$$\frac{\partial z}{\partial x} = 3x^2 - 6y$$

$$\frac{\partial z}{\partial y} = 24y^2 - 6x$$

$$A = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 6y) = 6x$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(24y^2 - 6x) = -6$$

$$C = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(24y^2 - 6x) = 48y$$

pa je:

$$D = A \cdot C - B^2$$

$$D = (6x)(48y) - (-6)^2$$

$$\boxed{D = 288xy - 36}$$

Uzimamo prvu stacionarnu tačku i ispitujemo:

$M_1(0,0)$

$$D = 288xy - 36 \rightarrow D(0,0) = 288 \cdot 0 \cdot 0 - 36 = -36 \rightarrow D(0,0) < 0$$

Dakle, pokazali smo da ova tačka nije ekstrem!

Ispitujemo drugu stacionarnu tačku:

$M_1(1, \frac{1}{2})$

$$D = 288xy - 36 \rightarrow D(1, \frac{1}{2}) = 288 \cdot 1 \cdot \frac{1}{2} - 36 = 144 - 36 = 108 \rightarrow D(1, \frac{1}{2}) > 0$$

Ova tačka jeste ekstrem, još da ispitamo da li je max ili min.

$$A = 6x \rightarrow A(1, \frac{1}{2}) = 6 \cdot 1 = 6 \rightarrow A(1, \frac{1}{2}) > 0$$

Zaključujemo da je ova tačka minimum!

Vratimo ovu vrednost u početnu funkciju da izračunamo tu minimalnu vrednost:

$$z = x^3 + 8y^3 - 6xy + 5$$

$$z_{\min}(1, \frac{1}{2}) = 1^3 + 8(\frac{1}{2})^3 - 6 \cdot 1 \cdot (\frac{1}{2}) + 5 = 1 + 1 - 3 + 5 = 4$$

$$\boxed{z_{\min}(1, \frac{1}{2}) = 4}$$

primer 2.

Ispitati ekstremume funkcije: $z = x\sqrt{y} - x^2 - y + 6x + 3$

Rešenje:

$$z = x\sqrt{y} - x^2 - y + 6x + 3$$
$$\frac{\partial z}{\partial x} = \sqrt{y} - 2x + 6$$
$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}} - 1$$

$$\sqrt{y} - 2x + 6 = 0$$
$$\frac{x}{2\sqrt{y}} - 1 = 0$$

Iz $\frac{x}{2\sqrt{y}} - 1 = 0 \to x = 2\sqrt{y}$ pa ovo zamenimo u $\sqrt{y} - 2x + 6 = 0$

$$\sqrt{y} - 2 \cdot 2\sqrt{y} + 6 = 0$$
$$-3\sqrt{y} + 6 = 0 \to \sqrt{y} = 2 \to \boxed{y = 4}$$
$$x = 2\sqrt{y} \to x = 2 \cdot \sqrt{4} \to \boxed{x = 4}$$

$\boxed{M(4,4)}$ je stacionarna tačka (jedina)

Dalje tražimo $A = \frac{\partial^2 z}{\partial x^2}$, $B = \frac{\partial^2 z}{\partial x \partial y}$, $C = \frac{\partial^2 z}{\partial y^2}$ i oformimo $D = A \cdot C - B^2$

$$\frac{\partial z}{\partial x} = \sqrt{y} - 2x + 6$$
$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}} - 1$$
$$A = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\sqrt{y} - 2x + 6) = -2$$
$$B = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{x}{2\sqrt{y}} - 1) = \frac{1}{2\sqrt{y}}$$
$$C = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{x}{2\sqrt{y}} - 1) = \frac{\partial}{\partial y}(\frac{x}{2} \cdot y^{-\frac{1}{2}} - 1) = -\frac{x}{4y\sqrt{y}}$$

$$D = A \cdot C - B^2$$

Pa je : $$D = (-2)(-\frac{x}{4y\sqrt{y}}) - (\frac{1}{2\sqrt{y}})^2$$

$$\boxed{D = \frac{x}{2y\sqrt{y}} - \frac{1}{4y}}$$

Vrednost stacionarne tačke $M(4,4)$ zamenimo u D :

$$D = \frac{x}{2y\sqrt{y}} - \frac{1}{4y}$$

$$D(4,4) = \frac{4}{2 \cdot 4\sqrt{4}} - \frac{1}{4 \cdot 4} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

$$D(4,4) = \frac{3}{16} > 0$$

Zaključujemo da je naša tačka $M(4,4)$ ekstrem, zamenimo je u A da odredimo da li je max ili min.

Kako je A = -2 nemamo šta da zamenjujemo , već odmah zaključujemo A = -2 < 0 , tačka $M(4,4)$ je maksimum!

Vratimo se u početnu funkciju da nađemo tu maksimalnu vrednost:

$$z = x\sqrt{y} - x^2 - y + 6x + 3$$

$$z(4,4) = 4\sqrt{4} - 4^2 - 4 + 6 \cdot 4 + 3$$

$$\boxed{z(4,4) = 15}$$

primer 3.

Ispitati ekstremume funkcije: $z = 3\ln\frac{x}{6} + 2\ln y + \ln(12 - x - y)$

<u>Rešenje:</u>

$$z = 3\ln\frac{x}{6} + 2\ln y + \ln(12 - x - y)$$

$$\frac{\partial z}{\partial x} = 3 \cdot \frac{1}{\frac{x}{6}} \cdot \frac{1}{6} + \frac{1}{12 - x - y}(-1) \rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{3}{x} - \frac{1}{12 - x - y}}$$

$$\frac{\partial z}{\partial y} = 2 \cdot \frac{1}{y} + \frac{1}{12 - x - y}(-1) \rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{2}{y} - \frac{1}{12 - x - y}}$$

Izjednačimo prve parcijalne izvode sa nulom i rešavamo sistem jednačina:

$$\frac{3}{x}-\frac{1}{12-x-y}=0$$
$$\frac{2}{y}-\frac{1}{12-x-y}=0$$

$$\frac{3}{x}=\frac{1}{12-x-y}\rightarrow 4x+3y=36$$
$$\frac{2}{y}=\frac{1}{12-x-y}\rightarrow 2x+3y=24$$

$$4x+3y=36$$
$$2x+3y=24$$

$$x=6$$
$$y=4$$
$$\boxed{M(6,4)}$$

Dobili smo stacionarnu tačku.

Tražimo $A=\frac{\partial^2 z}{\partial x^2}$, $B=\frac{\partial^2 z}{\partial x \partial y}$, $C=\frac{\partial^2 z}{\partial y^2}$ i oformimo $D=A\cdot C-B^2$

$$\frac{\partial z}{\partial x}=\frac{3}{x}-\frac{1}{12-x-y}$$
$$\frac{\partial z}{\partial y}=\frac{2}{y}-\frac{1}{12-x-y}$$

$$A=\frac{\partial^2 z}{\partial x^2}=-\frac{3}{x^2}-\frac{1}{(12-x-y)^2}$$
$$B=\frac{\partial^2 z}{\partial x \partial y}=-\frac{1}{(12-x-y)^2}$$
$$C=\frac{\partial^2 z}{\partial y^2}=-\frac{2}{y^2}-\frac{1}{(12-x-y)^2}$$

$$D=A\cdot C-B^2$$
$$D=(-\frac{3}{x^2}-\frac{1}{(12-x-y)^2})\cdot(-\frac{2}{y^2}-\frac{1}{(12-x-y)^2})-\frac{1}{(12-x-y)^4}$$

Zamenimo vrednost stacionarne tačke u D da vidimo da li je pna potencijalni ekstrem:

$$M(6,4) \to D = (-\frac{3}{x^2} - \frac{1}{(12-x-y)^2}) \cdot (-\frac{2}{y^2} - \frac{1}{(12-x-y)^2}) - \frac{1}{(12-x-y)^4}$$

$$D(6,4) = (-\frac{3}{6^2} - \frac{1}{(12-6-4)^2}) \cdot (-\frac{2}{4^2} - \frac{1}{(12-6-4)^2}) - \frac{1}{(12-6-4)^4}$$

$$D(6,4) = \frac{1}{8} > 0$$

Sad menjamo u A da odredimo da li je max ili min:

$$M(6,4) \to A = -\frac{3}{x^2} - \frac{1}{(12-x-y)^2}$$

$$A(6,4) = -\frac{3}{6^2} - \frac{1}{(12-6-4)^2} = -\frac{1}{12} - \frac{1}{4} < 0$$

Naša tačka M(6,4) je dakle maksimum!

Vrednost funkcije u njoj je:

$$z = 3\ln\frac{x}{6} + 2\ln y + \ln(12-x-y)$$

$$z(6,4) = 3\ln\frac{6}{6} + 2\ln 4 + \ln(12-6-4)$$

$$z(6,4) = 3\ln 1 + 2\ln 2^2 + \ln 2 = 3 \cdot 0 + 2 \cdot 2\ln 2 + \ln 2$$

$$\boxed{z(6,4) = 5\ln 2}$$

Ako imamo funkciju $u = u(x, y, z)$ postupak je sličan , ali imamo malo više posla....

Najpre tražimo:

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ i izjednačavamo sa 0 da bi našli **stacionarne tačke**.

Dalje tražimo:

$A = \frac{\partial^2 u}{\partial x^2}, B = \frac{\partial^2 u}{\partial x \partial y}, C = \frac{\partial^2 u}{\partial x \partial z}, D = \frac{\partial^2 u}{\partial y^2}, E = \frac{\partial^2 u}{\partial y \partial z}, F = \frac{\partial^2 u}{\partial z^2}$

Za svaku stacionarnu tačku radimo posebno:

Oformimo matricu:

$$A = \frac{\partial^2 u}{\partial x^2}(x_0, y_0, z_0), B = \frac{\partial^2 u}{\partial x \partial y}(x_0, y_0, z_0), C = \frac{\partial^2 u}{\partial x \partial z}(x_0, y_0, z_0), D = \frac{\partial^2 u}{\partial y^2}(x_0, y_0, z_0), E = \frac{\partial^2 u}{\partial y \partial z}(x_0, y_0, z_0), F = \frac{\partial^2 u}{\partial z^2}(x_0, y_0, z_0)$$

$$\begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2}(x_0, y_0, z_0) & \frac{\partial^2 u}{\partial x \partial y}(x_0, y_0, z_0) & \frac{\partial^2 u}{\partial x \partial z}(x_0, y_0, z_0) \\ \frac{\partial^2 u}{\partial x \partial y}(x_0, y_0, z_0) & \frac{\partial^2 u}{\partial y^2}(x_0, y_0, z_0) & \frac{\partial^2 u}{\partial y \partial z}(x_0, y_0, z_0) \\ \frac{\partial^2 u}{\partial x \partial z}(x_0, y_0, z_0) & \frac{\partial^2 u}{\partial y \partial z}(x_0, y_0, z_0) & \frac{\partial^2 u}{\partial z^2}(x_0, y_0, z_0) \end{pmatrix}$$

Tačka (x_0, y_0, z_0) je **lokalni minimum** funkcije ako je $d^2 u > 0$ ili ako tako neće, preko matrice mora da važi:

$$A > 0,\ \det\begin{pmatrix} A & B \\ B & D \end{pmatrix} > 0,\ \det\begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} > 0$$

Tačka (x_0, y_0, z_0) je **lokalni maksimum** funkcije ako je $d^2 u < 0$ ili ako tako neće, preko matrice mora da važi:

$$A < 0,\ \det\begin{pmatrix} A & B \\ B & D \end{pmatrix} > 0,\ \det\begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} < 0$$

primer 4.

Ispitati ekstremume funkcije $u = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}, \quad x > 0, y > 0, z > 0$

Rešenje:

Tražimo prve parcijalne izvode:

$$\frac{\partial u}{\partial x} = 1 + \frac{y^2}{4} \cdot \left(-\frac{1}{x^2}\right) = \boxed{1 - \frac{y^2}{4x^2}}$$
$$\frac{\partial u}{\partial y} = \frac{1}{4x} \cdot 2y + z^2 \cdot \left(-\frac{1}{y^2}\right) = \boxed{\frac{y}{2x} - \frac{z^2}{y^2}}$$
$$\frac{\partial u}{\partial z} = \boxed{\frac{2z}{y} - \frac{2}{z^2}}$$

Sad ih izjednačavamo sa nulom da bi našli stacionarne tačke, al vodimo računa da je $x > 0, y > 0, z > 0$.

$$1 - \frac{y^2}{4x^2} = 0 \rightarrow y^2 = 4x^2 \rightarrow y = \pm 2x \rightarrow \boxed{y = 2x} \text{ jer } x, y > 0 \text{ po zadatku}$$
$$\frac{y}{2x} - \frac{z^2}{y^2} = 0 \rightarrow y^3 = 2xz^2$$
$$\frac{2z}{y} - \frac{2}{z^2} = 0 \rightarrow y = z^3$$

$y = 2x$ zamenimo u $y^3 = 2xz^2$

$$(2x)^3 = 2xz^2 \rightarrow 8x^3 - 2xz^2 = 0 \rightarrow 2x(4x^2 - z^2) = 0 \rightarrow 4x^2 - z^2 = 0 \rightarrow \boxed{z = 2x}$$

Odavde je dakle $y = z \rightarrow$ zamenimo u $y = z^3 \rightarrow y = z = 1 \rightarrow x = \frac{1}{2}$

Dobili smo stacionarnu tačku $\left(\frac{1}{2}, 1, 1\right)$

Sad tražimo druge parcijalne izvode:

Prepišemo najpre prve parcijalne izvode, da bi nam bilo lakše za rad:

$$\frac{\partial u}{\partial x} = \boxed{1 - \frac{y^2}{4x^2}} \qquad \frac{\partial u}{\partial y} = \boxed{\frac{y}{2x} - \frac{z^2}{y^2}} \qquad \frac{\partial u}{\partial z} = \boxed{\frac{2z}{y} - \frac{2}{z^2}}$$

$$A = \frac{\partial^2 u}{\partial x^2} = \frac{y^2}{2x^3}$$

$$B = \frac{\partial^2 u}{\partial x \partial y} = -\frac{y}{2x^2}$$

$$C = \frac{\partial^2 u}{\partial x \partial z} = 0$$

$$D = \frac{\partial^2 u}{\partial y^2} = \frac{1}{2x} + \frac{2z^2}{y^3}$$

$$E = \frac{\partial^2 u}{\partial z \partial y} = -\frac{2z}{y^2}$$

$$F = \frac{\partial^2 u}{\partial z^2} = \frac{2}{y} + \frac{4}{z^3}$$

Sad vrednosti stacionarne tačke $\left(\frac{1}{2}, 1, 1\right)$ zamenjujemo u druge parcijalne izvode:

$$A = \frac{\partial^2 u}{\partial x^2} = \frac{y^2}{2x^3} \to A = \frac{1}{2 \cdot \frac{1}{8}} \to \boxed{A = 4}$$

$$B = \frac{\partial^2 u}{\partial x \partial y} = -\frac{y}{2x^2} \to B = -\frac{1}{2 \cdot \frac{1}{4}} \to \boxed{B = -2}$$

$$C = \frac{\partial^2 u}{\partial x \partial z} = 0 \to \boxed{C = 0}$$

$$D = \frac{\partial^2 u}{\partial y^2} = \frac{1}{2x} + \frac{2z^2}{y^3} \to D = 1 + 2 = 3 \to \boxed{D = 3}$$

$$E = \frac{\partial^2 u}{\partial z \partial y} = -\frac{2z}{y^2} \to \boxed{E = -2}$$

$$F = \frac{\partial^2 u}{\partial z^2} = \frac{2}{y} + \frac{4}{z^3} \to F = 2 + 4 = 6 \to \boxed{F = 6}$$

Da se podsetimo:

Tačka (x_0, y_0, z_0) je **lokalni minimum** funkcije ako je $d^2u > 0$ ili ako tako neće, preko matrice mora da važi:

$$A > 0, \ \det\begin{pmatrix} A & B \\ B & D \end{pmatrix} > 0, \ \det\begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} > 0$$

Da sve lepo proverimo:

A=4 > 0

$$\det\begin{pmatrix} A & B \\ B & D \end{pmatrix} = \begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} = 12 - 4 = 8 > 0$$

$$\det\begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{vmatrix} = 4 \cdot \begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix} + 2 \cdot \begin{vmatrix} -2 & -2 \\ 0 & 6 \end{vmatrix} = 4 \cdot 14 + 2 \cdot (-12) = 56 - 24 = 32 > 0$$

Zaključujemo da je tačka $\left(\frac{1}{2},1,1\right)$ lokalni minimum.

Još da nadjemo vrednost funkcije u toj tački:

$$u = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}, \quad x > 0, y > 0, z > 0$$

$$u\left(\frac{1}{2},1,1\right) = \frac{1}{2} + \frac{1^2}{4\frac{1}{2}} + \frac{1^2}{1} + \frac{2}{1} = \frac{1}{2} + \frac{1}{2} + 1 + 2 = 4$$

$$\boxed{u\left(\frac{1}{2},1,1\right) = 4}$$