INTEGRALI ZADACI (I – DEO)

Ako je f(x) neprekidna funkcija i **F \`(x) = f(x)** onda je $\boxed{\int f(x)dx = F(x) + C}$, gde je C proizvoljna konstanta.

Morate naučiti tablicu osnovnih integrala:

1. $\int dx = x + C$
2. $\int xdx = \frac{x^2}{2} + C$
3. $\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C}$ najčešće se koristi...
4. $\int \frac{1}{x} dx = \ln|x| + C$ ili da vas ne zbuni $\int \frac{dx}{x} = \ln|x| + C$
5. $\int a^x dx = \frac{a^x}{\ln a} + C$
6. $\int e^x dx = e^x + C$
7. $\int \sin xdx = -\cos x + C$
8. $\int \cos xdx = \sin x + C$
9. $\int \frac{1}{\sin^2 x} dx = -ctgx + C$
10. $\int \frac{1}{\cos^2 x} dx = tgx + C$
11. $\int \frac{1}{1+x^2} dx = \begin{matrix} arctgx + C \quad \text{ili} \\ -arcctgx + C \end{matrix}$ to jest $\boxed{\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} arctg \frac{x}{a} + C}$
12. $\int \frac{1}{\sqrt{1-x^2}} dx = \begin{matrix} \arcsin x + C \quad \text{ili} \\ -arccocx + C \end{matrix}$ to jest $\boxed{\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C}$

Ovo su osnovni tablični integrali. Neki profesori dozvoljavaju da se kao tablični koriste i :

13. $\int \frac{dx}{1-x^2} = \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right| + C$ odnosno $\boxed{\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C}$ to jest $\boxed{\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C}$
14. $\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln\left|x + \sqrt{x^2 \pm 1}\right| + C$ odnosno $\boxed{\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C}$

Primeri:

1. $\int x^5 dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$ kao 3. tablični $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

2. $\int 7^x dx = \frac{7^x}{\ln 7} + C$ kao 5. tablični $\int a^x dx = \frac{a^x}{\ln a} + C$

3. $\int \sqrt{x} dx =$ pogledamo i vidimo da ga ovaj integral nema u tablici osnovnih integrala... Ideja je da se kod ovakvih integrala iskoristi pravilo za stepenovanje $\boxed{\sqrt[m]{x^n} = x^{\frac{n}{m}}}$, odnosno $\boxed{\sqrt{x} = x^{\frac{1}{2}}}$. Na ovaj način se integral svede na najčešće upotrebljavani tablični $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

Dakle:

$$\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2x^{\frac{3}{2}}}{3} + C$$

4. $\int \frac{1}{x^{12}} dx =$ I ovaj ga nema u tablici...Za njega ćemo upotrebiti pravilo za stepenovanje, da je $\frac{1}{x^n} = x^{-n}$ ...

$$\int \frac{1}{x^{12}} dx = \int x^{-12} dx = \frac{x^{-12+1}}{-12+1} + C = \frac{x^{-11}}{-11} + C = \frac{1}{-11x^{11}} + C$$

Najbolje je da se mi podsetimo svih pravila za stepenovanje i korenovanje:

1) $a^0 = 1$

2) $a^{-n} = \frac{1}{a^n}$

3) $a^m \cdot a^n = a^{m+n}$

4) $a^m : a^n = a^{m-n}$

5) $(a^m)^n = a^{m \cdot n}$

6) $(a \cdot b) = a^n \cdot b^n$

7) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

8) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

1) $\sqrt[n]{a^m} = a^{\frac{m}{n}}$
2) $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
3) $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{a : b}$
4) $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$
5) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a}$
6) $\sqrt[n\cancel{p}]{a^{m\cancel{p}}} = \sqrt[n]{a^m}$

Nastavimo sa primerima…

5. $\int x \cdot \sqrt[3]{x} dx =$ Upotrebimo pravila za stepen i koren da " pripremimo" podintegralnu funkciju :

$$x \cdot \sqrt[3]{x} = x^1 \cdot x^{\frac{1}{3}} = x^{1+\frac{1}{3}} = x^{\frac{4}{3}}$$

$$\int x \cdot \sqrt[3]{x} dx = \int x^{\frac{4}{3}} dx = \frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} + C = \frac{x^{\frac{7}{3}}}{\frac{7}{3}} + C = \frac{3x^{\frac{7}{3}}}{7} + C$$

6. $$\underline{\int 5^x \cdot 3^{-x} dx} = \int 5^x \cdot \frac{1}{3^x} dx = \int \left(\frac{5}{3}\right)^x dx = \frac{\left(\frac{5}{3}\right)^x}{\ln\left(\frac{5}{3}\right)} + C$$

7.

$$\int \sqrt{x\sqrt{x\sqrt{x}}} dx = ?$$

"Spakujemo" podintegralnu funkciju $\sqrt{x\sqrt{x\sqrt{x}}} = \sqrt{x\sqrt{\sqrt{x^2 \cdot x}}} = \sqrt{x\sqrt[4]{x^3}} = \sqrt{\sqrt[4]{x^4 \cdot x^3}} = \sqrt[8]{x^7} = x^{\frac{7}{8}}$

$$\int x^{\frac{7}{8}} dx = \frac{x^{\frac{7}{8}+1}}{\frac{7}{8}+1} + C = \frac{x^{\frac{15}{8}}}{\frac{15}{8}} + C = \frac{8x^{\frac{15}{8}}}{15} + C$$

Da se upoznamo i sa osnovnim svojstvima neodredjenog integrala:

1) $\boxed{\int A \cdot f(x) dx = A \cdot \int f(x) dx}$ gde je A konstanta (broj)

Dakle, slično kao i kod izvoda, konstanta (broj) izlazi ispred integrala...

Primeri:

8.

$$\int 4x^3dx = ?$$

$$\int 4x^3dx = 4\int x^3dx = \cancel{4}\frac{x^4}{\cancel{4}} + C = x^4 + C$$

9.

$$\int \frac{1}{4x}dx = ?$$

$$\int \frac{1}{4x}dx = \frac{1}{4}\cdot\int \frac{1}{x}dx = \frac{1}{4}\cdot\ln|x| + C$$

10.

$$\int 2\pi\sin xdx = ?$$

$$\int 2\pi\sin xdx = 2\pi\cdot\int \sin xdx = 2\pi\cdot(-\cos x) + C = -2\pi\cos x + C$$

2) $\boxed{\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx}$

Opet slično kao kod izvoda: Ako imamo zbir ili razliku više funkcija od svake tražimo posebno integral...

11.

$$\int (4x^2 + 2x - 3)dx = ?$$

$\int (4x^2 + 2x - 3)dx = \int 4x^2dx + \int 2xdx - \int 3dx =$ konstante izbacimo ispred integrala...

$$= 4\int x^2dx + 2\int xdx - 3\int dx$$

$$= 4\frac{x^3}{3} + 2\frac{x^2}{2} - 3x + C = \boxed{4\frac{x^3}{3} + x^2 - 3x + C}$$

12.

$$\int (5\cos x + \frac{1}{3}e^x - 2x^3 + \frac{4}{x} - \frac{23}{\sin^2 x} + 2\cdot 5^x)dx = ?$$

$$\int (5\cos x + \frac{1}{3}e^x - 2x^3 + \frac{4}{x} - \frac{23}{\sin^2 x} + 2\cdot 5^x)dx = 5\int \cos xdx + \frac{1}{3}\int e^xdx - 2\int x^3dx + 4\int \frac{1}{x}dx - 23\int \frac{dx}{\sin^2 x} + 2\cdot\int 5^x dx =$$

$$= 5\sin x + \frac{1}{3}e^x - 2\frac{x^4}{4} + 4\ln|x| - 23(-ctgx) + 2\frac{5^x}{\ln 5} + C$$

13.

$$\int \frac{x-2}{x^3}dx = ?$$

Kod ovog i sličnih integrala ćemo upotrebiti $\boxed{\frac{A \pm B}{C} = \frac{A}{C} \pm \frac{B}{C}}$

$$\int \frac{x-2}{x^3}dx = \int (\frac{x}{x^3} - \frac{2}{x^3})dx = \int (x^{-2} - 2x^{-3})dx = \int x^{-2}dx - 2\int x^{-3}dx$$

$$= \frac{x^{-2+1}}{-2+1} - 2\frac{x^{-3+1}}{-3+1} + C$$

$$= \boxed{-\frac{1}{x} + \frac{1}{x^2} + C}$$

14.

$$\int \frac{2^{x+1} - 5^{x-1}}{10^x}dx = ?$$

Da prisredimo najpre malo podintegralnu funkciju...

$$\frac{2^{x+1} - 5^{x-1}}{10^x} = \frac{2^x \cdot 2^1 - \frac{5^x}{5^1}}{10^x} = \frac{2^x \cdot 2^1}{10^x} - \frac{\frac{5^x}{5^1}}{10^x} = 2 \cdot \left(\frac{2}{10}\right)^x - \frac{1}{5} \cdot \left(\frac{5}{10}\right)^x = 2 \cdot \left(\frac{1}{5}\right)^x - \frac{1}{5} \cdot \left(\frac{1}{2}\right)^x$$

$$\int \frac{2^{x+1} - 5^{x-1}}{10^x}dx = \int [2 \cdot \left(\frac{1}{5}\right)^x - \frac{1}{5} \cdot \left(\frac{1}{2}\right)^x]dx = \int 2 \cdot \left(\frac{1}{5}\right)^x dx - \int \frac{1}{5} \cdot \left(\frac{1}{2}\right)^x dx =$$

$$= 2 \cdot \int \left(\frac{1}{5}\right)^x dx - \frac{1}{5} \cdot \int \left(\frac{1}{2}\right)^x dx = 2 \cdot \frac{\left(\frac{1}{5}\right)^x}{\ln \frac{1}{5}} - \frac{1}{5}\frac{\left(\frac{1}{2}\right)^x}{\ln \frac{1}{2}} + C$$

15.

$\int \frac{x^2}{x^2+1}dx = ?$ Ovo je tip integrala koji najlakše rešavamo malim "trikom" (dodamo 1 i oduzmemo 1)

$$\int \frac{x^2}{x^2+1}dx = \int \frac{x^2+1-1}{x^2+1}dx = \int \frac{x^2+1}{x^2+1}dx - \int \frac{1}{x^2+1}dx = \int \frac{\cancel{x^2+1}}{\cancel{x^2+1}}dx - \int \frac{1}{x^2+1}dx$$

$$= \int dx - \int \frac{1}{x^2+1}dx = \boxed{x - arctgx + C}$$

I ovde će nam trebati znanje iz trigonometrije. Da se podsetimo nekih najvažnijih formula:

Osnovni trigonometrijski indetiteti

1) $\sin^2\alpha + \cos^2\alpha = 1$

2) $tg\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

3) $ctg\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

4) $tg\alpha \cdot ctg\alpha = 1$

Adicione formule

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$tg(\alpha+\beta) = \frac{tg\alpha + tg\beta}{1 - tg\alpha \cdot tg\beta}$$
$$ctg(\alpha+\beta) = \frac{ctg\alpha \cdot ctg\beta - 1}{ctg\beta + ctg\alpha}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$tg(\alpha-\beta) = \frac{tg\alpha - tg\beta}{1 + tg\alpha \cdot tg\beta}$$
$$ctg(\alpha-\beta) = \frac{ctg\alpha \cdot ctg\beta + 1}{ctg\beta - ctg\alpha}$$

Polovina ugla

1. $\sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$ ili $2\sin^2\dfrac{\alpha}{2} = 1-\cos\alpha$ odnosno $\boxed{\sin^2 x = \dfrac{1-\cos 2x}{2}}$
2. $\cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$ ili $2\cos^2\dfrac{\alpha}{2} = 1+\cos\alpha$ odnosno $\boxed{\cos^2 x = \dfrac{1+\cos 2x}{2}}$
3. $tg\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$
4. $ctg\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}$

Transformacije zbira i razlike u proizvod

1. $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$
2. $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$
3. $\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$
4. $\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$
5. $tg\alpha \pm tg\beta = \dfrac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$
6. $ctg\alpha \pm ctg\beta = \dfrac{\sin(\alpha\pm\beta)}{\sin\alpha\sin\beta}$

Dvostruki ugao

1. $\sin 2\alpha = 2\sin\alpha\cos\alpha$
2. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
3. $tg 2\alpha = \dfrac{2tg\alpha}{1-tg^2\alpha}$
4. $ctg 2\alpha = \dfrac{ctg^2\alpha - 1}{2ctg\alpha}$

16.

$$\int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x} dx = ?$$ treba nam formula $\boxed{\cos 2x = \cos^2 x - \sin^2 x}$

$$\int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x} dx = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cdot \cos^2 x} dx =$$ rastavimo na dva integrala...

$$\int \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x} dx - \int \frac{\sin^2 x}{\sin^2 x \cdot \cos^2 x} dx = skratimo...$$

$$\int \frac{\cancel{\cos^2 x}}{\sin^2 x \cdot \cancel{\cos^2 x}} dx - \int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x} \cdot \cos^2 x} dx =$$ i dobijamo dva tablična integrala...

$$\int \frac{1}{\sin^2 x} dx - \int \frac{1}{\cos^2 x} dx = \boxed{-ctgx - tgx + C}$$

17.

$$\int \frac{dx}{\sin^2 x \cdot \cos^2 x} = ?$$ treba nam formula $\boxed{\sin^2 x + \cos^2 x = 1}$

$$\int \frac{1 \cdot dx}{\sin^2 x \cdot \cos^2 x} = \text{uz dx je 1, zar ne?} = \int \frac{1}{\sin^2 x \cdot \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx =$$ rastavimo na dva integrala...

$$\int \frac{\sin^2 x}{\sin^2 x \cdot \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x} dx = skratimo...$$

$$\int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x} \cdot \cos^2 x} dx + \int \frac{\cancel{\cos^2 x}}{\sin^2 x \cdot \cancel{\cos^2 x}} dx =$$ i dobijamo dva tablična integrala...

$$\int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx = \boxed{tgx - ctgx + C}$$

18.

$$\int tg^2 x dx = ?$$

Ovde koristimo $tgx = \dfrac{\sin x}{\cos x}$

$$\int tg^2 x dx = \int \frac{\sin^2 x}{\cos^2 x} dx =$$ kako je $\sin^2 \alpha + \cos^2 \alpha = 1 \to \boxed{\sin^2 \alpha = 1 - \cos^2 \alpha}$, pa je

$$= \int \frac{1 - \cos^2 \alpha}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx - \int \frac{\cancel{\cos^2 \alpha}}{\cancel{\cos^2 x}} dx = \int \frac{1}{\cos^2 x} dx - \int dx = \boxed{tgx - x + C}$$

19.

$$\int \frac{18x^2-2}{3x-1}dx = ?$$

Probamo da sredimo podintegralnu funkciju, ako neće, mora se koristiti neki drugi trik(druga metoda)...

$$\frac{18x^2-2}{3x-1} = \frac{2(9x^2-1)}{3x-1} = \frac{2\cancel{(3x-1)}(3x+1)}{\cancel{3x-1}} = 2(3x+1) = 6x+2$$ Sad je već lakše...

$$\int \frac{18x^2-2}{3x-1}dx = \int (6x+2)dx = 6\int xdx + 2\int dx = 6\frac{x^2}{2} + 2x + C = \boxed{3x^2+2x+C}$$

20.

$$\int \frac{4-x}{2+\sqrt{x}}dx = ?$$

$$\frac{4-x}{2+\sqrt{x}} = \frac{2^2-(\sqrt{x})^2}{2+\sqrt{x}} = \frac{(2-\sqrt{x})(2+\sqrt{x})}{2+\sqrt{x}} = \frac{(2-\sqrt{x})\cancel{(2+\sqrt{x})}}{\cancel{2+\sqrt{x}}} = 2-\sqrt{x} = 2-x^{\frac{1}{2}}$$

$$\int \frac{4-x}{2+\sqrt{x}}dx = \int (2-x^{\frac{1}{2}})dx = \int 2dx - \int x^{\frac{1}{2}}dx = 2x - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \boxed{2x - \frac{2x^{\frac{3}{2}}}{3} + C}$$