POWER SERIES (II – PART)

Example 1.

Function $f(x) = arctgx$ develop into a power series and determine its interval of convergence.

Solution:

The idea is to use familiar development $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad -1<x<1$

So, here is $x^2 \in (-1,1) \to x \in (-1,1)$

How to apply theorem $\int_a^b (\sum_{n=0}^{\infty} a_n x^n)dx = \sum_{n=0}^{\infty}(\int_a^b a_n x^n dx)$

$$f(x) = arctgx = \int_0^x \frac{1}{1+x^2}dx = \int_0^x (\sum_{n=0}^{\infty}(-1)^n x^{2n})dx = \sum_{n=0}^{\infty}(-1)^n \boxed{\int_0^x x^{2n}dx} = \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}}$$

You should always examine the convergence of the resulting series on the borders of the interval of convergence.

In our case it is : x = -1 and x = 1

For x = -1

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \to for\ x=-1 \to \sum_{n=0}^{\infty}(-1)^n \frac{(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n \overbrace{\boxed{(-1)^{2n}}}^{\text{this is } 1}(-1)^1}{2n+1} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}}$$

This is an alternative series , and it will apply the Leibniz criteria:

$$\lim_{n\to\infty}\frac{1}{2n+1} = 0 \quad \text{and we have : } n+1>n \to 2(n+1)>2n \to 2(n+1)+1>2n+1 \to \frac{1}{2(n+1)+1} < \frac{1}{2n+1} \to \boxed{a_{n+1} < a_n}$$

This means that the alternative series is convergent

For x = 1

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \to For\ x=1 \to \sum_{n=0}^{\infty}(-1)^n \frac{(1)^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}}$$

Similar to the previous series and here, number series is convergent (Leibniz criterion)

Hence, our power series is convergent for $x \in [-1,1]$

Example 2.

Function $f(x) = \ln\frac{2+x}{1-x}$ develop into a power series.

Solution:

$$f(x) = \ln\frac{2+x}{1-x}$$

$$f`(x) = \frac{1}{\frac{2+x}{1-x}} \cdot \left(\frac{2+x}{1-x}\right)` = \frac{1-x}{2+x} \cdot \frac{1(1-x)+1(2+x)}{(1-x)^2} = \frac{3}{(2+x)(1-x)}$$

We got a rational function:

$$\frac{3}{(2+x)(1-x)} = \frac{A}{2+x} + \frac{B}{1-x} \quad .... / \cdot (2+x)(1-x)$$

$$3 = A(1-x) + B(2+x)$$

$$3 = A - Ax + 2B + Bx$$

$$3 = x(-A+B) + A + 2B$$

$$-A + B = 0$$

$$\underline{A + 2B = 3}$$

$$3B = 3 \rightarrow \boxed{B=1} \rightarrow \boxed{A=1}$$

$$\frac{3}{(2+x)(1-x)} = \frac{1}{2+x} + \frac{1}{1-x} = \frac{1}{2(1+\frac{x}{2})} + \frac{1}{1-x} = \boxed{\frac{1}{2} \cdot \frac{1}{(1+\frac{x}{2})} + \frac{1}{1-x}}$$

We will use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad -1<x<1$

$$\frac{1}{1+\frac{x}{2}} = \frac{1}{1-(-\frac{x}{2})} = \sum_{n=0}^{\infty} (-\frac{x}{2})^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^n$$

The radius of convergence of this series is:

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{2^n}}{\frac{1}{2^{n+1}}} = \lim_{n\to\infty} \frac{2^n \cdot 2}{2^n} = 2 \text{ , so: } \quad x \in (-2, 2)$$

Now, for the interval $x \in (-1,1)$ (Which belongs to a given interval $(-2,2)$) we have :

$$f`(x)=\frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}}+\frac{1}{1-x}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n+\sum_{n=0}^{\infty}\ x^n=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2\cdot 2^n}x^n+\sum_{n=0}^{\infty}\ x^n$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n+\sum_{n=0}^{\infty}\ x^n$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n+2^{n+1}}{2^{n+1}}x^n}$$

Now go back through the integral to function f (x):

$$f(x)=\int_0^x(\sum_{n=0}^{\infty}\frac{(-1)^n+2^{n+1}}{2^{n+1}}x^n)dx=\sum_{n=0}^{\infty}\frac{(-1)^n+2^{n+1}}{2^{n+1}}\int_0^x x^n d=\sum_{n=0}^{\infty}\frac{(-1)^n+2^{n+1}}{2^{n+1}}\cdot\frac{x^{n+1}}{n+1}=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n+2^{n+1}}{2^{n+1}(n+1)}x^{n+1}}$$

<u>Example 3.</u>

Function $f(x)=\frac{1+x}{(1-x)^3}$, develop into a power series and then determine the sum $\sum_{n=1}^{\infty}\frac{n^2}{2^{n-1}}$

<u>Solution :</u>

Again we "separate" the function.

$$\frac{1+x}{(1-x)^3}=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{(1-x)^3}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots/\cdot(1-x)^3$$

$$1+x=A(1-x)^2+B(1-x)+C$$

$$1+x=A(1-2x+x^2+B-Bx+C$$

$$1+x=A-2Ax+Ax^2+B-Bx+C$$

$$1+x=+Ax^2+x(-2A-B)+A+B+C$$

$$A=0$$

$$-2A-B=1$$

$$\underline{A+B+C=1}$$

$$B=-1\rightarrow C=2$$

$$\frac{1+x}{(1-x)^3}=\frac{0}{1-x}+\frac{-1}{(1-x)^2}+\frac{2}{(1-x)^3}=\frac{2}{(1-x)^3}-\frac{1}{(1-x)^2}$$

$So:$

$$\boxed{f(x)=\frac{2}{(1-x)^3}-\frac{1}{(1-x)^2}}$$

We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad -1<x<1$

Mark with $g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Derivative is :

$$g`(x) = -\frac{1}{(1-x)^2}(1-x)` = -\frac{1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$$

$$g``(x) = -\frac{1}{(1-x)^4}((1-x)^2)` = -\frac{1}{(1-x)^4}2(1-x)(-1) = \frac{2}{(1-x)^3}$$

Now we have :

$$\frac{1}{(1-x)^2} = g`(x) = \left(\sum_{n=0}^{\infty} x^n\right)^{/} = \sum_{n=1}^{\infty} nx^{n-1}$$ Watch out: we must change that n goes from 1.

$$\frac{2}{(1-x)^3} = g``(x) = \left(g`(x)\right)` = \left(\sum_{n=1}^{\infty} nx^{n-1}\right)^{/} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

Watch out: we must change that n goes from 2, but since the previous sum goes is 1, we will do a small correction for the second sum , we will put that goes from 1, and where we have *n ,* we write *n +1*

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=1}^{\infty}(n+1)(n+1-1)x^{n+1-2} = \sum_{n=1}^{\infty}(n+1)nx^{n-1} = \boxed{\sum_{n=1}^{\infty} n(n+1)x^{n-1}}$$

Now we return to the task:

$$f(x) = g``(x) - g`(x) = \sum_{n=1}^{\infty} n(n+1)x^{n-1} - \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty}[n(n+1)-n]x^{n-1} = \sum_{n=1}^{\infty}[n^2+n-n]x^{n-1}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}}$$

To find the required sum $\sum_{n=1}^{\infty}\frac{n^2}{2^{n-1}}$, we need instead of x in $f(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}$ to put some number.

Here it is obvious that it should be $\frac{1}{2}$. This value change in the initial function :

$$f(x) = \frac{1+x}{(1-x)^3} \rightarrow f(\frac{1}{2}) = \frac{1+\frac{1}{2}}{(1-\frac{1}{2})^3} = 12$$

Example 4.

Determine the area of convergence and the sum $\sum_{n=1}^{\infty} n^2 x^n$ and then find the sum of the numerical series $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2^n}$

Solution :

As is $a_n = n^2$ we will use $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ **=R**

$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \lim_{n\to\infty}\frac{n^2}{n^2+2n+1} = \frac{1}{1} = 1$ **, so we have** R=1

Series is convergent for $x \in (-1,1)$. We must examine what happens for x = -1 and for x = 1

For x = 1

We have numerical series $\sum_{n=1}^{\infty} n^2 1^{n-1} = \sum_{n=1}^{\infty} n^2$. As is $\lim_{n\to\infty} n^2 = \infty$**, he diverges.**

For x = -1

Similar thinking: $\sum_{n=1}^{\infty} n^2 (-1)^{n-1}$, he diverges because general is not the approaches zero.

We conclude that the area of convergence remains $x \in (-1,1)$

We will use well-known development $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ -1<x<1.

Make a small correction :($\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ all multiply by x)

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \qquad x \in (-1,1)$$

Further work:

$$\sum_{n=1}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} n \cdot n \cdot \underbrace{\boxed{x}}_{go\ ahead} \cdot x^{n-1} = x\sum_{n=1}^{\infty} n \cdot \underbrace{\boxed{n \cdot x^{n-1}}}_{\substack{derivative\\ of\ x^n}} = x\sum_{n=1}^{\infty} n \cdot (x^n)` = x\left(\sum_{n=1}^{\infty} n \cdot x^n\right)' =$$

we take x and front brackets :

$$x\left(\sum_{n=1}^{\infty} n \cdot x^n\right)' = x\left(x\sum_{n=1}^{\infty} n \cdot x^{n-1}\right)' = x\left(x\sum_{n=1}^{\infty} (x^n)`\right)' = x\left(x\left(\underbrace{\sum_{n=1}^{\infty} x^n}_{replace\ this}\right)'\right)' = x\left(x\left(\frac{x}{1-x}\right)'\right)'$$

Now we have a job to find derivatives:

$$x\left(x\left(\frac{x}{1-x}\right)'\right)' = x\left(x\left(\frac{1-x+x}{(1-x)^2}\right)\right)' = x\left(\frac{x}{(1-x)^2}\right)' = x \cdot \frac{(1-x)^2 - 2(1-x)(-1)x}{(1-x)^4} =$$

$$= x\frac{(1-x)^2 + 2(1-x)x}{(1-x)^4} = x\frac{\cancel{(1-x)}[1-x+2x]}{(1-x)^{\cancel{4}}} = x\frac{x+1}{(1-x)^3} = \boxed{\frac{x(x+1)}{(1-x)^3}}$$

Sum of number series $\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{2^n}$ we will find when we put $-\frac{1}{2}$ in $\sum_{n=1}^{\infty} n^2 x^n$ or $\frac{x(x+1)}{(1-x)^3}$ instead of x .

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{2^n} = \frac{-\frac{1}{2}(-\frac{1}{2}+1)}{(1-(-\frac{1}{2}))^3} = \frac{-\frac{1}{4}}{\frac{27}{8}} = \boxed{-\frac{2}{27}}$$

Example 5.

Examine the convergence $\sum_{n=1}^{\infty}\frac{n+1}{n}x^n$ and find its sum.

Solution :

From $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ **=R** we have : $\lim_{n\to\infty}\frac{\frac{n+1}{n}}{\frac{n+2}{n+1}} = \lim_{n\to\infty}\frac{(n+1)^2}{n(n+2)} = \lim_{n\to\infty}\frac{n^2+2n+1}{n^2+2n} = \frac{1}{1} = 1$

So: $x \in (-1,1)$.

For x = - 1

We get series $\sum_{n=1}^{\infty}\frac{n+1}{n}(-1)^n$, it is obvious that $\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$ **series diverges.**

For x = 1

A similar situation : For $\sum_{n=1}^{\infty}\frac{n+1}{n}$ is $\lim_{n\to\infty}\frac{n+1}{n}=1\neq 0$ **then series diverges.**

We conclude that the area of convergence of saeries is interval $(-1,1)$.

Mark : $f(x)=\sum_{n=1}^{\infty}\frac{n+1}{n}x^n$

First, we will, at the interval of convergence, integrate serie , to "destroy" n +1

$$\int_0^x f(x)dx=\int_0^x\left(\sum_{n=1}^{\infty}\frac{n+1}{n}x^n\right)dx=\sum_{n=1}^{\infty}\frac{n+1}{n}\int_o^x x^n dx=\sum_{n=1}^{\infty}\frac{\cancel{n+1}}{n}\frac{x^{n+1}}{\cancel{n+1}}=\sum_{n=1}^{\infty}\frac{x^{n+1}}{n}$$

Throw x in front: $\sum_{n=1}^{\infty}\frac{x^{n+1}}{n}=x\sum_{n=1}^{\infty}\frac{x^n}{n}$

From here is:

$$\int_0^x f(x)dx=x\sum_{n=1}^{\infty}\frac{x^n}{n}\text{.......................................}/:x$$

$$\frac{1}{x}\int_0^x f(x)dx=\sum_{n=1}^{\infty}\frac{x^n}{n}$$

Now look for derivative from this:

$$\frac{1}{x}\int_0^x f(x)dx=\sum_{n=1}^{\infty}\frac{x^n}{n}\to\left(\frac{1}{x}\int_0^x f(x)dx\right)^{\prime}=\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)^{\prime}=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n}=\sum_{n=1}^{\infty}x^{n-1}=\frac{1}{1-x}$$

We got $\left(\frac{1}{x}\int_0^x f(x)dx\right)^{\prime}=\frac{1}{1-x}$

To find $\frac{1}{x}\int_0^x f(x)dx$ we must integrate $\frac{1}{1-x}$:

$$\frac{1}{x}\int_0^x f(x)dx=\int_0^x\frac{1}{1-x}dx=-\ln|1-x|\Big|_0^x=\boxed{-\ln|1-x|}$$

From here we have :

$$\frac{1}{x}\int_0^x f(x)dx = -\ln|1-x| \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots / *x$$

$$\int_0^x f(x)dx = -x\ln|1-x|$$

Finally will be:

$$f(x) = \left(\int_0^x f(x)dx\right)' = \underbrace{\left(-x\ln|1-x|\right)'}_{derivative\ products} = -1\cdot\ln|1-x| + \frac{1}{1-x}(-1)(-x) = \boxed{-\ln(1-x) + \frac{x}{1-x}}$$