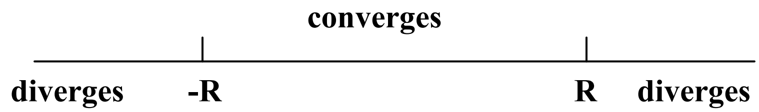
POWER SERIES (I – PART)

DEF: $\sum_{n=0}^{\infty} a_n \mathbf{(t\text{-}t_0)^n}$ is power series, if we put **$t-t_0 = x$**, we have $\sum_{n=0}^{\infty} a_n x^n = a_0+a_1x+\ldots+a_nx^n+\ldots$

Partial sum is $S_n(x)=\sum_{k=0}^{n} a_k x^k$; n-th rest is $R_n(x)= \sum_{k=0}^{\infty} a_{n+k} x^{n+k}$

If there is an R so that $|x|<R$ then the series converges , and for $|x|>R$ diverges.

Interval (-R,R) is the interval of convergence



For x=R and x=-R, working separately, using the criteria for the convergence of number series.

The number R is called the RADIUS OF CONVERGENCE of the power series.

Cauchy formula: $\overline{\lim_{n\to\infty}} \left|\frac{a_n}{a_{n+1}}\right| = R$

Root formula : $\frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}} = R$ or $\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$

Apply the following theorem: Let $S(x)= \sum_{n=0}^{\infty} a_n x^n$

1) $\lim_{x\to x_0} S(x)= \lim_{x\to x_0} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (\lim_{x\to x_0} a_n x^n)=S(x_0)$

2) $\int_a^b (\sum_{n=0}^{\infty} a_n x^n)dx = \sum_{n=0}^{\infty} (\int_a^b a_n x^n dx)$

3) **Power series in the interval of convergence can be differentiated by the member**

DEVELOPMENT

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ where is $(-\infty<x<\infty)$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $(-\infty<x<\infty)$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, $(-\infty<x<\infty)$

$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$, $-1<x<1$

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$, $-1<x<1$

$\ln(1-x)=- \sum_{n=1}^{\infty} \frac{x^n}{n}$, $-1<x<1$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $-1<x<1$

$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n$

Example 1.

Determine the radius of convergence and examine the convergence of the ends of the interval of convergence for the following power series:

a) $\sum_{n=0}^{\infty}(n+1)x^n$

b) $\sum_{n=0}^{\infty}\frac{x^n}{n}$

c) $\sum_{n=0}^{\infty}\frac{2^n x^n}{n^2+1}$

Solutions:

a)

$$\sum_{n=0}^{\infty}(n+1)x^n$$

Here is $a_n = n+1$ and we will use **:** $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ **=R**

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{n+1}{n+2} = \frac{1}{1} = 1$$

We got that series converges in the interval $(-1,1)$. Now we must examine for x = -1 and x = 1

For x = -1

Substituting this value in the given series : $\sum_{n=0}^{\infty}(n+1)x^n \to \sum_{n=0}^{\infty}(n+1)(-1)^n$

We have obtained an alternative series . As is $\lim_{n\to\infty}(n+1) = \infty$ we conclude here that the series diverges.

For x = 1

$$\sum_{n=0}^{\infty}(n+1)x^n \to \sum_{n=0}^{\infty}(n+1)(1)^n = \boxed{\sum_{n=0}^{\infty}(n+1)}$$

The number series also diverges, because: $\lim_{n\to\infty}(n+1) = \infty$

Conclusion: $\sum_{n=0}^{\infty}(n+1)x^n$ is convergent on the interval $(-1,1)$

b) $\sum_{n=0}^{\infty} \frac{x^n}{n}$

Here is $a_n = \frac{1}{n}$ so it is convenient to again use the Cauchy formula:

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{1} = 1$$

R=1 , and for now we have that series converges on the interval (- 1,1)

<u>For x = -1</u>

$$\sum_{n=0}^{\infty} \frac{x^n}{n} \to \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

We have obtained an alternative series where is $a_n = \frac{1}{n}$

Series is decreasing because $n < n+1 \to \frac{1}{n} > \frac{1}{n+1} \to a_n > a_{n+1}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. Leibniz criterion : converge here!

<u>For x = 1</u>

$$\sum_{n=0}^{\infty} \frac{x^n}{n} \to \sum_{n=0}^{\infty} \frac{1^n}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$$

For this series since before we know that diverges (see previous files on a number series)

Conclusion:

Series $\sum_{n=0}^{\infty} \frac{x^n}{n}$ is convergent on the interval [-1,1)

c) $\sum_{n=0}^{\infty} \frac{2^n x^n}{n^2+1}$

As is $a_n = \frac{2^n}{n^2+1}$ It is convenient to try Cauchy formula:

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\frac{2^n}{n^2+1}}{\frac{2^{n+1}}{(n+1)^2+1}} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \cdot \frac{(n+1)^2+1}{n^2+1} = \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{2^n} \cdot 2} \cdot \underbrace{\boxed{\frac{n^2+2n+2}{n^2+1}}}_{\text{This is } 1} = \frac{1}{2}$$

This means that order converges, for now, in the interval $(-\frac{1}{2}, \frac{1}{2})$

For $x = -\frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{2^n (-\frac{1}{2})^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{2^n \frac{(-1)^n}{2^n}}{n^2+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

Here is $a_n = \frac{1}{n^2+1}$.

$\lim_{n\to\infty} \frac{1}{n^2+1} = 0$ Leibniz criterion : this series converges.

For $x = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{2^n (\frac{1}{2})^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{2^n \frac{1}{2^n}}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$$

The number series is also convergent!

Conclusion: Series is convergent on the interval $\sum_{n=0}^{\infty} \frac{2^n x^n}{n^2+1}$ $[-\frac{1}{2}, \frac{1}{2}]$

Example 2.

Determine the radius of convergence and examine the convergence of the ends of the interval of convergence for the following power series:

a) $\sum_{n=0}^{\infty}\left(\frac{n+1}{n}\right)^{n^2} x^n$

b) $\sum_{n=0}^{\infty}(-2)^n x^{2n}$

Rešenje:

Here we use another formula to find radius of convergence: $\frac{1}{R}=\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}$

a) $\sum_{n=0}^{\infty}\left(\frac{n+1}{n}\right)^{n^2} x^n$

$$\frac{1}{R}=\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}=\overline{\lim_{n\to\infty}}\sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2}}=\overline{\lim_{n\to\infty}}\left(\frac{n+1}{n}\right)^{n}=\overline{\lim_{n\to\infty}}\left(1+\frac{1}{n}\right)^{n}=e$$

$$\frac{1}{R}=e\to\boxed{R=\frac{1}{e}}$$

So now we know that this series converges in the interval $(-\frac{1}{e},\frac{1}{e})$.

For $x=\frac{1}{e}$

$$\sum_{n=0}^{\infty}\left(\frac{n+1}{n}\right)^{n^2}(\frac{1}{e})^n=\sum_{n=0}^{\infty}\left(\frac{n+1}{n}\right)^{n^2}\frac{1}{e^n}$$

Check first whether the general approaches zero:

$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n^2}\frac{1}{e^n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n\cdot n}\frac{1}{e^n}=\lim_{n\to\infty}e^n\frac{1}{e^n}=1$ From this we conclude that the series diverges.

For $x = -\frac{1}{e}$

Here is a alternative series , but similar ways of thinking come to the conclusion that the series diverges here.

Conclusion: $\sum_{n=0}^{\infty}\left(\frac{n+1}{n}\right)^{n^2} x^n$ converges in the interval $(-\frac{1}{e},\frac{1}{e})$.

b) $\sum_{n=0}^{\infty}(-2)^n x^{2n}$

We will use same criteria:

$$\frac{1}{R} = \overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|} = \overline{\lim_{n\to\infty}}\sqrt[n]{2^n} = \overline{\lim_{n\to\infty}} 2 = 2$$

$$\frac{1}{R} = 2 \to \boxed{R = \frac{1}{2}}$$

We went to watch:

Let's look at a given series $\sum_{n=0}^{\infty}(-2)^n x^{2n} = \sum_{n=0}^{\infty}(-2)^n \left(x^2\right)^n$

This means that this refers to the radius of convergence x^2 and for x will be :

$R = \frac{1}{2}$ is for $x^2 \to R = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ is for x

Series therefore converges in the interval $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

For $x = -\frac{1}{\sqrt{2}}$ and $x = \frac{1}{\sqrt{2}}$ Series will be divergent because obviously the general not approaches zero.

Conclusion: Series $\sum_{n=0}^{\infty}(-2)^n x^{2n}$ converges in the interval $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.