Partial differential equations

1. Determine Cauchy solution for partial differential equations:

$$y^2 p + yzq + z^2 = 0$$

that satisfy conditions: x – y = 0 and x – yz = 1

Solution:

$y^2 p + yzq + z^2 = 0$ First, we have z^2 to switch to the other side!

$y^2 p + yzq = -z^2$ Now make the system differential equations in a symmetrical form

$\frac{dx}{y^2} = \frac{dy}{yz} = \frac{dz}{-z^2}$ Look at second and third member of this equality…

$\frac{dy}{yz} = \frac{dz}{-z^2}$ Multiply all with z

$\frac{dy}{y} = \frac{dz}{-z}$ Now integral

$\int \frac{dy}{y} = -\int \frac{dz}{z}$ from here is $\ln|y| = -\ln|z| + \ln|c_1| \longrightarrow y = \frac{c_1}{z} \longrightarrow c_1 = yz$

So, the first first integral is $\psi_1 = yz$

Find now the second first integral:

From $c_1 = yz$ express $z = \frac{c_1}{y}$ and from equality $\frac{dx}{y^2} = \frac{dy}{yz} = \frac{dz}{-z^2}$ is:

$\frac{dx}{y^2} = \frac{dy}{yz}$ here we first all multiply with y and then replace z with $z = \frac{c_1}{y}$,

$\frac{dx}{y} = \frac{dy}{z} \longrightarrow \frac{dx}{y} = \frac{dy}{\frac{c_1}{y}} \longrightarrow c_1 dx = y^2 dy$: integral

$c_1 x = \frac{y^3}{3} + c_2 \xleftarrow{\text{replace}} c_1 = yz$

$yzx = \frac{y^3}{3} + c_2$ express here the constant: $c_2 = yzx - \frac{y^3}{3}$

We have the second first integral: $\psi_2 = yzx - \frac{y^3}{3}$

Solutions are : $\psi_1 = yz$ **and** $\psi_2 = yzx - \frac{y^3}{3}$

Whether they are good solutions?

We need to examine their independence! And must be true:

$$\frac{D(\psi_1, \psi_2)}{D(x, y)} \neq 0 \longrightarrow \begin{vmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_2}{\partial x} \\ \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_2}{\partial y} \end{vmatrix} \neq 0 \quad \text{For our task is} \quad \begin{vmatrix} 0 & yz \\ z & zx - y^2 \end{vmatrix} \neq 0 \quad \textbf{is!}$$

Solutions are good!

Further we solve Cauchy task: **x – y = 0** **and** **x – yz = 1**

What to do here?

$\psi_1 = yz$ **and** $\psi_2 = yzx - \frac{y^3}{3}$ and the conditions **x – y = 0** **and** **x – yz = 1, all these we use to eliminate unknowns and find a connection between the solutions.**

How is **x – yz = 1** and $\psi_1 = yz$ **must be** $x - \overline{\psi_1} = 1$ so: $x = \overline{\psi_1} + 1$

x – y = 0 $\longrightarrow$ x = y = $\overline{\psi_1} + 1$

$$\psi_2 = yzx - \frac{y^3}{3} \longrightarrow \overline{\psi_2} = \overline{\psi_1}(1 + \overline{\psi_1}) - \frac{(1 + \overline{\psi_1})^3}{3}$$

So we find a connection between the solutions and we eliminate unknowns x, y and z

In $\overline{\psi_2} = \overline{\psi_1}(1 + \overline{\psi_1}) - \frac{(1 + \overline{\psi_1})^3}{3}$ we will return the right values: $\psi_1 = yz$ **,** $\psi_2 = yzx - \frac{y^3}{3}$

$\overline{\psi_2} = \overline{\psi_1}(1 + \overline{\psi_1}) - \frac{(1 + \overline{\psi_1})^3}{3}$ all multiply with 3

$3\overline{\psi_2} = 3\overline{\psi_1}(1 + \overline{\psi_1}) - (1 + \overline{\psi_1})^3$ here change $\psi_1 = yz$ **,** $\psi_2 = yzx - \frac{y^3}{3}$ instead $\overline{\psi_1}$ and $\overline{\psi_2}$

$3(xyz - \frac{y^3}{3}) = 3yz(1 + yz) - (1 + yz)^3$ **simplify little…….and**

$3xyz - y^3 + 1 + y^3 z^3 = 0$ **is the final solutions**

2. Determine Cauchy solution for partial differential equations:

$$yp + xq = x^2 + y^2$$

that satisfy conditions: : x = 1 and z = 1 + 2y +3 y^2

Solution:

$yp + xq = x^2 + y^2$ go to the symmetrical system

$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{x^2 + y^2}$ From here, select the first two members of equality

$\frac{dx}{y} = \frac{dy}{x}$ ⟶ xdx = ydy Integral

$\int xdx = \int ydy$ So: $\frac{x^2}{2} = \frac{y^2}{2} + c_1*$ (here as a small "trick" take c_1*) All multiply with 2…

$x^2 = y^2 + 2\, c_1*$ and $2\, c_1* = c_1$ then is $x^2 = y^2 + c_1$ or

$c_1 = x^2 - y^2$ ⟶ $\psi_1 = x^2 - y^2$ **the first first integral**

Find now the second first integral

$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{x^2 + y^2}$ Add to the first member of equality up and down **y**, add to second member of equality up and down x

$\frac{ydx}{y^2} = \frac{xdy}{x^2} = \frac{dz}{x^2 + y^2}$ Gather first two members of equality

$$\frac{ydx + xdy}{y^2 + x^2} = \frac{dz}{x^2 + y^2} \quad \text{then} \quad \frac{d(xy)}{y^2 + x^2} = \frac{dz}{x^2 + y^2} \longrightarrow d(xy)=dz \quad \text{Integral...}$$

$xy = z + c_2$ so: $\psi_2 = xy - z$ **is the second first integral**

$\psi_1 = x^2 - y^2$ **and** $\psi_2 = xy - z$ are the first integrals, test their independence:

$$\begin{vmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_2}{\partial x} \\ \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_2}{\partial y} \end{vmatrix} \neq 0 \qquad \begin{vmatrix} 2x & y \\ -2y & x \end{vmatrix} \neq 0$$ **means that solutions are good!**

Cauchy task **: x = 1 and z = 1 + 2y +3 y^2**

First, in both solutions replace x = 1:

$\overline{\psi_1} = 1 - y^2$ **and** $\overline{\psi_2} = y - z$ **from here is** $1 - \overline{\psi_1} = y^2 \longrightarrow y = \sqrt{1 - \overline{\psi_1}}$ **and** $y - \overline{\psi_2} = z$

Furthermore, this change in

z = 1 + 2y +3 y^2

$y - \overline{\psi_2}$ **= 1 + 2y + 3** $(1 - \overline{\psi_1})$

3$\overline{\psi_1} - \overline{\psi_2}$ **- 4 = y**

3$\overline{\psi_1} - \overline{\psi_2}$ **- 4 =** $\sqrt{1 - \overline{\psi_1}}$ **here now change solutions** $\psi_1 = x^2 - y^2$ **and** $\psi_2 = xy - z$ instead $\overline{\psi_1}$ and $\overline{\psi_2}$

$3(x^2 - y^2)$ - $(xy - z)$ - 4 = $\sqrt{1 - (x^2 - y^2)}$ **simplify little…**

final solution is: $z = 4 - 3x^2 + 3y^2 + xy + \sqrt{1 - (x^2 - y^2)}$

3. Find the general solution of partial differential equations:

$$\mathbf{xp + yq = z - xy}$$

Solution:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - xy}$$

$$\frac{dx}{x} = \frac{dy}{y}$$ integral

$$\int \frac{dx}{x} = \int \frac{dy}{y} \longrightarrow \ln|x| = \ln|y| + \ln|c_1| \longrightarrow x = y\,c_1 \longrightarrow c_1 = \frac{x}{y}, \quad \text{so:}$$

$\psi_1 = \frac{x}{y}$ is **the first first integral**

From $x = y\,c_1$ is $y = \frac{x}{c_1}$ and from $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - xy}$ we'll take the first and third member.

$\frac{dx}{x} = \frac{dz}{z - xy}$ replace that $y = \frac{x}{c_1}$ and we have:

$$\frac{dx}{x} = \frac{dz}{z - x\frac{x}{c_1}} \longrightarrow \frac{dx}{x} = \frac{dz}{z - \frac{x^2}{c_1}}$$ simplify little...

$$\frac{dz}{dx} = \frac{z}{x} - \frac{x}{c_1} \longrightarrow z` = \frac{z}{x} - \frac{x}{c_1} \longrightarrow z` - \frac{z}{x} = -\frac{x}{c_1}$$ linear d.e.

$$z` - \frac{z}{x} = -\frac{x}{c_1}$$

$$z(x) = e^{-\int p(x)dx}\left(c_2 + \int q(x) e^{\int p(x)dx} dx\right)$$

$$\int p(x)dx = -\int \frac{1}{x}dx = -\ln|x| = \ln|x|^{-1}$$

$$\int q(x) e^{\int p(x)dx} dx = -\int \frac{x}{c_1} e^{\ln x^{-1}} dx = -\int \frac{1}{c_1} dx = -\frac{x}{c_1}$$

$$z(x) = x\left(c_2 - \frac{x}{c_1}\right) \longleftarrow c_1 = \frac{x}{y} \quad \text{so:}$$

$z = x(c_2 - y)$ and express here the constant $c_2 = y + \frac{z}{x}$

$\psi_2 = y + \frac{z}{x}$ **is the second first integral**

Check independence of solutions:

$$\begin{vmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_2}{\partial x} \\ \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_2}{\partial y} \end{vmatrix} \neq 0 \qquad \begin{vmatrix} \frac{1}{y} & \frac{-z}{x^2} \\ \frac{-x}{y^2} & 1 \end{vmatrix} \neq 0$$ **Obviously is!**

So:

$\psi_1 = \frac{x}{y}$ **the first first integral**

$\psi_2 = y + \frac{z}{x}$ **the second first integral**

Important: When you find firsts integrales general solution we can write in the form of F(ψ_1, ψ_2)=0

So, in our case would be : $F(\frac{x}{y}, y + \frac{z}{x}) = 0$

More is that if z comes only in one of the first integrals, the general solution we can write in the form of:

$\psi_1 =$ **f(ψ_2) if z occurs in the** ψ_1 and

$\psi_2 =$ **f(ψ_1) if z occurs in the** ψ_2

In our case, z occurs in ψ_2 **and the solution, we can write as:**

$y + \frac{z}{x} = \mathbf{f}(\frac{x}{y})$ and from here we can express z, if necessary…

$\frac{z}{x} = \mathbf{f}(\frac{x}{y}) - y$ when all multiply with x…

$z = x\,\mathbf{f}(\frac{x}{y}) - xy$

4. Find the integrated curve of partial differential equations :

$$yz\frac{\partial z}{\partial x}+zx\frac{\partial z}{\partial y}+2xy=0$$

which passes through circle $x^2+y^2=16$ for $z=3$

Solution:

$yz\frac{\partial z}{\partial x}+zx\frac{\partial z}{\partial y}+2xy=0$ we know that $\frac{\partial z}{\partial x}=p\wedge\frac{\partial z}{\partial y}=q$

$yzp+zxq=-2xy$

$\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{-2xy}$

$\frac{dx}{yz}=\frac{dy}{zx}$ all multiply with z

$\frac{dx}{y}=\frac{dy}{x}$ from here is $\int xdx=\int ydy$ then $\frac{x^2}{2}=\frac{y^2}{2}+c_1{}^*$ $\longrightarrow$ $\mathbf{x^2=y^2+c_1}$ where is $\mathbf{c_1=2c_1{}^*}$

$c_1=x^2-y^2$ $\longrightarrow$ $\psi_1=x^2-y^2$ **is the first first integral**

Let's go back now in the initial system:

$\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{-2xy}$ expand the first member of equality with x, and second with y

$\frac{xdx}{xyz}=\frac{ydy}{yzx}=\frac{dz}{-2xy}$ gather now the first two members of equality

$\frac{xdx+ydy}{2xyz}=\frac{dz}{-2xy}$ multiply all with 2xy

$\frac{xdx+ydy}{z}=\frac{dz}{-1}$ multiply all with z and we have

xdx + ydy = -z dz integral

$\frac{x^2}{2}+\frac{y^2}{2}=-\frac{z^2}{2}+c_2^*$ multiply all with 2

$x^2+y^2=-z^2+2c_2{}^*$ we'll mark $2c_2{}^*=c_2$

$x^2+y^2=-z^2+c_2$ $\longrightarrow$ $x^2+y^2+z^2=c_2$

$\psi_2 = \mathbf{x^2 + y^2 + z^2}$ **is the second first integral**

$\psi_1 = x^2 - y^2$ **is the first first integral**

Check independence: $\begin{vmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_2}{\partial x} \\ \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_2}{\partial y} \end{vmatrix} \neq 0$ $\quad \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix} = 8xy \neq 0$

Now $x^2 + y^2 = 16$ for $z = 3$

Change these values in $\psi_2 = \mathbf{x^2 + y^2 + z^2}$ **So:** $\overline{\psi_2} = 16 + 3^2 = 16 + 9 = 25$, **Conclude:**

$\mathbf{x^2 + y^2 + z^2 = 25}$

is requested integral curve, this is a sphere (central) with a radius of $r = 5$

5. Find the general solution of partial differential equations:

$$(2z - 3y)\frac{\partial u}{\partial x} + (3x - z)\frac{\partial u}{\partial y} + (y - 2x)\frac{\partial u}{\partial z} = 0$$

Solution:

$\frac{dx}{2z - 3y} = \frac{dy}{3x - z} = \frac{dz}{y - 2x}$ multiply with 2 second member of equality

$\frac{dx}{2z - 3y} = \frac{2dy}{6x - 2z} = \frac{dz}{y - 2x}$ gather now the first two members of equality

$\frac{dx + 2dy}{6x - 3y} = \frac{dz}{y - 2x}$ simplify little...

$\frac{dx + 2dy}{3(2x - y)} = \frac{-dz}{2x - y}$ all multiply with 3(2x-y)

dx + 2dy = -3 dz integral

$x + 2y = -3z + c_1$ ⟶ $c_1 = x + 2y + 3z$

$\psi_1 = x + 2y + 3z$ **is the first first integral**

Let's go back to the start system:

$$\frac{dx}{2z-3y}=\frac{dy}{3x-z}=\frac{dz}{y-2x}$$ "expand" the first, second and third member of equality with x, y, z

$$\frac{xdx}{2xz-3xy}=\frac{ydy}{3xy-yz}=\frac{zdz}{yz-2xz}$$ gather the first two members of equality

$$\frac{xdx+ydy}{2xz-yz}=\frac{zdz}{yz-2xz}$$

$$\frac{xdx+ydy}{2xz-yz}=\frac{-zdz}{2xz-yz}$$

$xdx + ydy = - zdz$ integral

$$\frac{x^2}{2}+\frac{y^2}{2}=-\frac{z^2}{2}+c_2^*$$ multiply all with 2

$x^2 + y^2 + z^2 = c_2$ where is: $c_2 = 2c_2^*$

$\psi_2 = x^2 + y^2 + z^2$ is the second first integral

Finally solution is: u = f($x+2y+3z$, $x^2 + y^2 + z^2$)

Where f is arbitrary integrable functions.