Systems of differential equations (examples)

Normal form

1. Solve the system of equations:

$$\frac{dx}{dt} = -7x + y$$
$$\frac{dy}{dt} = -2x - 2y$$

Solution:

What is the idea of these tasks?

For one of the equation we find derivate and replace that in the other equation.

We need to make to stay only one unknown!

$$\frac{dx}{dt} = -7x + y \qquad \frac{dy}{dt} = -2x - 5y$$

first introduce labels x` and y` to be easier to work , of course: x` = x`(t) and y` = y` (t)

$$x` = -7x + y \qquad y` = -2x - 5y$$

from the first equation express y = x` + 7x and find derivate

$$x`` = -7x` + y` \qquad y` = -2x - 5y$$

Now, y` replace in first equation , also y = x` + 7x

$$x`` = -7x` + (-2x - 5y) = -7x` - 2x - 5y = -7x` - 2x - 5(x` + 7x) = -7x` - 2x - 5x` - 35x = -12x` - 37x$$

$$x`` = -12x` - 37x$$

$x`` + 12x` + 37x = 0$ First,we solve the characteristic equation…

$$\lambda^2 + 12\lambda + 37 = 0$$

$$\lambda_{1,2} = \frac{-12 \pm 2i}{2} \Rightarrow \lambda_1 = -6 + i, \lambda_2 = -6 - i$$

To remind you a little theory from this part…

Linear homogeneous equation with constant coefficient (second order)

$$y'' + a_1 y' + a_2 = 0$$

First, write down the **characteristic equation:**

$$\lambda^2 + a_1\lambda + a_2 = 0$$

Depending on the characteristic equation solutions, we have three differentiate cases:

1) λ_1 and λ_2 are real and different, it is: $y(x)=c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$

2) λ_1 and λ_2 are real and equal solutions, it is: $y(x)= c_1e^{\lambda_1 x} + x\,c_2e^{\lambda_2 x}$

3) λ_1 and λ_2 are complex conjugate: λ_1=a+bi, λ_2=a-bi ,then: $y(x)=c_1e^{ax}\cos bx+c_2e^{ax}\sin bx$

Since our solutions are $\lambda_1 = -6+i, \lambda_2 = -6-i$, it is obvious that a = -6 and b = 1, so, solution is:

$$x_t = c_1e^{-6t}\cos t + c_2e^{-6t}\sin t$$

To find now y_t. We have already expressed y = x` + 7x …

$$x_t = c_1e^{-6t}\cos t + c_2e^{-6t}\sin t$$

$$x`_t = c_1(-6e^{-6t}\cos t - e^{-6t}\sin t) + c_2(-6e^{-6t}\sin t + e^{-6t}\cos t)$$ **this replace in y = x` + 7x**

$$y_t = c_1(-6e^{-6t}\cos t - e^{-6t}\sin t) + c_2(-6e^{-6t}\sin t + e^{-6t}\cos t) + 7(c_1e^{-6t}\cos t + c_2e^{-6t}\sin t)$$

$$y_t = -c_1 6e^{-6t}\cos t - c_1e^{-6t}\sin t - c_2 6e^{-6t}\sin t + c_2e^{-6t}\cos t + 7c_1e^{-6t}\cos t + 7c_2e^{-6t}\sin t$$

$$y_t = c_1e^{-6t}\cos t - c_1e^{-6t}\sin t + c_2e^{-6t}\sin t + c_2e^{-6t}\cos t$$

$$y_t = c_1(e^{-6t}\cos t - e^{-6t}\sin t) + c_2(e^{-6t}\sin t + e^{-6t}\cos t)$$

Therefore, the final solution is:

$$x_t = c_1e^{-6t}\cos t + c_2e^{-6t}\sin t$$
$$y_t = c_1(e^{-6t}\cos t - e^{-6t}\sin t) + c_2(e^{-6t}\sin t + e^{-6t}\cos t)$$

2. Solve the system of equations:

$$\frac{dx}{dt} = x + 2y + t$$
$$\frac{dy}{dt} = 2x + y + t$$

Solution:

$x` = x + 2y + t$
$y` = 2x + y + t$ do not forget: x` = x`(t) and y` = y` (t)

$x` = x + 2y + t$ ⟶ x`` = x` + 2y` + 1 (when we find derivate)

$x` = x + 2y + t \Rightarrow 2y = x` - x - t \Rightarrow y = \frac{x` - x - t}{2}$

x`` = x` + 2y` + 1

$x`` = x` + 2(x + y + t) + 1$

$x`` = x` + 2x + 2y + 2t + 1$

$x`` = x` + 2x + 2\frac{x` - x - t}{2} + 2t + 1$

$x`` - 2x` - 3x = t + 1$ **Linear nonhomogeneous equation with constant coefficient**

$x`` - 2x` - 3x = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$\lambda_{1,2} = \frac{2 \pm 4}{2} \Rightarrow \lambda_1 = -1, \lambda_2 = 3$ and homogeneous solution “by x” is equal:

$x_t(H) = c_1 e^{-t} + c_2 e^{3t}$

Method undetermined coefficients :

$$\left.\begin{array}{l} X = At + B \\ X` = A \\ X`` = 0 \end{array}\right\}$$ This replace in $x`` - 2x` - 3x = t + 1$

-2 A –3 (At+B) = t + 1

-2 A –3 At-3B = t + 1

-3At –2A-3B = t + 1   from here is -3A=1 and -2A-3B=1 so: $A = -\frac{1}{3}$ and $B = -\frac{1}{9}$

$X = -\frac{1}{3}t - \frac{1}{9}$

$x_t = c_1 e^{-t} + c_2 e^{3t} - \frac{t}{3} - \frac{1}{9}$ **is solution «by x»**

How is $y = \frac{x` - x - t}{2}$, we will find derivate from $x_t = c_1 e^{-t} + c_2 e^{3t} - \frac{t}{3} - \frac{1}{9}$ and replace that in y.

$$x_t = c_1 e^{-t} + c_2 e^{3t} - \frac{t}{3} - \frac{1}{9}$$

$$x`_t = -c_1 e^{-t} + 3c_2 e^{3t} - \frac{1}{3}$$

$$y_t = \frac{1}{2}[(-c_1 e^{-t} + 3c_2 e^{3t} - \frac{1}{3}) - (c_1 e^{-t} + c_2 e^{3t} - \frac{t}{3} - \frac{1}{9}) - t]$$

$$y_t = -c_1 e^{-t} + c_2 e^{3t} - \frac{t}{3} - \frac{1}{9}$$

Therefore, the final solution is:

$$x_t = c_1 e^{-t} + c_2 e^{3t} - \frac{t}{3} - \frac{1}{9}$$
$$y_t = -c_1 e^{-t} + c_2 e^{3t} - \frac{t}{3} - \frac{1}{9}$$

3. Solve the system of equations:

$$\frac{dy}{dx} = y + z + x$$
$$\frac{dz}{dx} = -4y - 3z + 2x$$

Solution:

$y` = y + z + x$
$z` = -4y - 3z + 2x$ **here is z=z(x) and y=y(x)**

Express **z** from the first equation $y` = y + z + x \Rightarrow y` - y - x = z$

Find derivate of first equation:

$y`` = y` + z` + 1$ here we will replace z` and z

$y`` = y` + (-4y - 3z + 2x) + 1 = y` - 4y - 3z + 2x + 1 = y` - 4y - 3(y` - y - x) + 2x + 1$

$y`` = y` - 4y - 3y` + 3y + 3x + 2x + 1$

$y`` + 2y` + y = 5x + 1$ **Linear nonhomogeneous equation…**

$y`` + 2y` + y = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda_{1,2} = \frac{-2 \pm 0}{2} \Rightarrow \lambda_1 = -1, \lambda_2 = -1$

$y_x(H) = c_1 e^{-x} + c_2 x e^{-x}$

Y= Ax+B
Y`= A
Y``= 0
this replace in $y`` + 2y` + y = 5x + 1$

$0 + 2A + Ax + B = 5x + 1$

$Ax + 2A + B = 5x + 1$ from here is **A = 5** and **2A+B = 1** ⟶ **A= 5** and **B = -9**

Y= Ax+B So: Y = 5x-9 , go back in **homogeneous solution** $y_x(H) = c_1 e^{-x} + c_2 x e^{-x}$

$y_x = c_1 e^{-x} + c_2 x e^{-x} + Y$

$y_x = c_1e^{-x} + c_2xe^{-x} + 5x - 9$ We have received a solution "by y", now to find "by z"…….

$y`_x = -c_1e^{-x} + c_2(e^{-x} - xe^{-x}) + 5$ replace in $z = y` - y - x$

$z_x = -c_1e^{-x} + c_2(e^{-x} - xe^{-x}) + 5 - (c_1e^{-x} + c_2xe^{-x} + 5x - 9) - x$

$z_x = (c_2 - 2c_1 - 2c_2x)e^{-x} - 6x + 14$ We have received a solution "by z"

Therefore, **the final solution is:**

$$y_x = c_1e^{-x} + c_2xe^{-x} + 5x - 9$$
$$z_x = (c_2 - 2c_1 - 2c_2x)e^{-x} - 6x + 14$$

4. Solve the system of equations:

$$\frac{dx}{dt} = y + z$$
$$\frac{dy}{dt} = x + z$$
$$\frac{dz}{dt} = x + y$$

Solution:

$x` = y + z$
$y` = x + z$ ⟶ **x` = x`(t), y` = y` (t) and z` = z`(t)**
$z` = x + y$

Find derivate of first equation: $x`` = y` + z`$ and replace that in y` and z` , so:

$x`` = y` + z`$ = (x +z) + (x + y) = 2x + **y + z,** and because $x` = y + z$ it will be

x`` = 2x + x` ⟶ x`` - x` - 2x = 0

x`` - x` - 2x = 0

$\lambda^2 - \lambda - 2 = 0$

$\lambda_{1,2} = \frac{1 \pm 3}{2} \Rightarrow \lambda_1 = -1, \lambda_2 = 2$ and we find solution «by x»:

$x_t = c_1e^{-t} + c_2e^{2t}$

Now looking for solutions by y and by z . Let's go back to the start system:

$$x' = y + z$$
$$y' = x + z$$
$$z' = x + y$$

\- **Subtract the first and third of the equation!**

z` - x` = x – z here we will replace $x_t = c_1 e^{-t} + c_2 e^{2t}$ and where find derivate from here ,we will have x`

$$x_t = c_1 e^{-t} + c_2 e^{2t}$$

$$x'_t = -c_1 e^{-t} + 2c_2 e^{2t}$$

z` - x` = x – z replace x and x`

$$z' - (-c_1 e^{-t} + 2c_2 e^{2t}) = c_1 e^{-t} + c_2 e^{2t} - z$$

z ` + z = $3c_2 e^{2t}$ **Linear differential equations by "z"**

$$z(t) = e^{-\int p(t)dt}(c_3 + \int q(t) e^{\int p(t)dt} dt)$$

$$\int p(t)dt = \int 1dt = t$$

$$z(t) = e^{-t}(c_3 + \int 3c_2 e^{2t} e^{t} dt)$$

$$z(t) = e^{-t}(c_3 + 3c_2 \int e^{3t} dt) = e^{-t}(c_3 + 3c_2 \frac{1}{3} e^{3t}) = e^{-t}(c_3 + c_2 e^{3t}) = e^{-t} c_3 + c_2 e^{2t}$$

So we provided a solution "by z": $z(t) = e^{-t} c_3 + c_2 e^{2t}$

More to find a solution "by y"!

$$y' = x + z = c_1 e^{-t} + c_2 e^{2t} + e^{-t} c_3 + c_2 e^{2t}$$

$$y' = (c_1 + c_3) e^{-t} + 2c_2 e^{2t}$$ integral

$$y_t = \int [(c_1 + c_3) e^{-t} + 2c_2 e^{2t}] dt = -(c_1 + c_3) e^{-t} + c_2 e^{2t}$$

So: $y_t = -(c_1 + c_3) e^{-t} + c_2 e^{2t}$

$$x_t = c_1 e^{-t} + c_2 e^{2t}$$
$$y_t = -(c_1 + c_3) e^{-t} + c_2 e^{2t}$$
$$z_t = e^{-t} c_3 + c_2 e^{2t}$$

the final solution!

5. Solve the system of equations: $\begin{array}{l} x` - 2x - 4y = \cos t \\ y` + x + 2y = \sin t \end{array}$ **and find a solution for x(0) = 4 and y(0) = 1**

Solution:

We will express y from the first equation:

$$x` - 2x - 4y = \cos t$$
$$x` - 2x - \cos t = 4y$$
$$y = \frac{x` - 2x - \cos t}{4}$$

Find derivate for $x` - 2x - 4y = \cos t$ **:**

$$x` - 2x - 4y = \cos t$$

$x`` - 2x` - 4y` = -\sin t$ here replace **y`** from start system

x`` - 2x` -4(sin t –x – 2y)= - sin t

x``- 2x` - 4 sint + 4x +8y = - sint replace **y** $y = \frac{x` - 2x - \cos t}{4}$

x``- 2x` - 4 sint + 4x +8 $\frac{x` - 2x - \cos t}{4}$ = - sin t

x`` = 3sin t + 2cos t

$$x` = \int (3\sin t + 2\cos t)dt = -3\cos t + 2\sin t + c_1$$

$$x` = -3\cos t + 2\sin t + c_1$$

$x = \int (-3\cos t + 2\sin t + c_1)dt = -3\sin t - 2\cos t + c_1 t + c_2$ **So, we find x_t**

$$x_t = -3\sin t - 2\cos t + c_1 t + c_2$$

To find y, we will go from $y = \frac{x` - 2x - \cos t}{4}$

$$y = \frac{x` - 2x - \cos t}{4} = \frac{1}{4}[(-3\cos t + 2\sin t + c_1) - 2(-3\sin t - 2\cos t + c_1 t + c_2) - \cos t]$$

$$y_t = \frac{1}{4}(8\sin t - 2c_1 t - 2c_2 + c_1)$$

We have received a general solution:

$$x_t = -3\sin t - 2\cos t + c_1 t + c_2$$
$$y_t = \frac{1}{4}(8\sin t - 2c_1 t - 2c_2 + c_1)$$

To find that it satisfies the **conditions: x(0) = 4 and y(0) = 1**

4 = $-3\sin 0 - 2\cos 0 + c_1 0 + c_2$ **here is obviously** $c_2=4$

1 = $\frac{1}{4}(8\sin 0 - 2c_1 0 - 2c_2 + c_1)$ $\longrightarrow$ $c_1 = 16$

Solution that satisfies the conditions is :

$$x_t = -3\sin t - 2\cos t + 16t + 4$$
$$y_t = 2\sin t - 8t + 1$$

Symmetrical form

1. **Find the first integrals for the system:**

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{-xy}$$

Solution:

We'll take the first two members of this equality:

$\frac{dx}{xz} = \frac{dy}{yz}$ obviously we can all multiply with z

$\frac{dx}{x} = \frac{dy}{y}$ integral

$$\int \frac{dx}{x} = \int \frac{dy}{y} \longrightarrow \ln|x| = \ln|y| + \ln|c_1| \longrightarrow \ln|x| = \ln|yc_1| \longrightarrow |x| = |yc_1|$$

x = y c_1 then $c_1 = \frac{x}{y}$ first the first integral.

So: $c_1 = \frac{x}{y}$ **is the first first integral.**

In the most tasks is not difficult to find the first first integral, but we try to find the second first integral, problems arise ...

But, we always have the option to express one unknown from obtained solutions and replace it in the home equation.

You can always try to use some "trick"……. for our example:

$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{-xy}$ **The idea is to the first member of equality add y up and down, and to another member add x**

$\frac{ydx}{yxz} = \frac{xdy}{xyz} = \frac{dz}{-xy}$ **Now gather first two members of equality**

$\frac{ydx + xdy}{2xyz} = \frac{dz}{-xy}$ **ydx + xdy we can write as:** **ydx + xdy = d(xy)**

$\frac{d(xy)}{2xyz} = \frac{dz}{-xy}$ **all multiply with xy**

$\frac{d(xy)}{2z} = \frac{dz}{-1}$ from here is **d(xy) = -2z dz** $\longrightarrow$ **xy = $-z^2 + c_2$** $\longrightarrow$ **$c_2 = xy + z^2$**

and this is required **the second first integral**

Solution is :

$c_1 = \frac{x}{y}$ **the first first integral**
$c_2 = xy + z^2$ **the second first integral**

These two relations defines general integral of the system!

2. Find the first integrals for the system:

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{y-x}$$

Solution:

Gather first two members of equality:

$$\frac{dx+dy}{y-x} = \frac{dz}{y-x} \qquad \text{all multiply with } (y-x)$$

$dx + dy = dz$ integral

$x + y = z + c_1 \longrightarrow$ **$c_1 = x + y - z$** **the first first integral**

Express from here $z = x + y - c_1$ and replace in :

$$\frac{dx}{y-z} = \frac{dy}{z-x}$$

$$\frac{dx}{y-(x+y-c_1)} = \frac{dy}{(x+y-c_1)-x}$$

$$\frac{dx}{-x-c_1} = \frac{dy}{y-c_1}$$

$$\frac{dy}{dx} = \frac{y-c_1}{-x-c_1} \longrightarrow y` = \frac{y-c_1}{-x-c_1} \longrightarrow y` = -\frac{y}{x+c_1} + \frac{c_1}{x+c_1}$$

$y` + \frac{y}{x+c_1} = \frac{c_1}{x+c_1}$ **this is a first-order linear differential equation, we have formula:**

$$y(x) = e^{-\int p(x)dx}(c_2 + \int q(x)e^{\int p(x)dx}dx)$$

$$\int p(x)dx = \int \frac{1}{x+c_1}dx = \ln|x+c_1|$$

$$y(x) = e^{-\int p(x)dx}(c_2 + \int q(x)e^{\int p(x)dx}dx) = e^{-\ln|x+c_1|}(c_2 + \int \frac{c_1}{x+c_1}(x+c_1)dx) = \frac{1}{x+c_1}(c_2 + c_1 x) \qquad \text{So:}$$

$y = \frac{1}{x+c_1}(c_2 + c_1 x)$ in here we replace **$c_1 = x + y - z$**

y = $\frac{1}{x + x + y - z}(c_2 + x(x + y - z))$

y = $\frac{c_2 + x^2 + xy - xz}{2x + y - z}$

$2xy + y^2 - yz = c_2 + x^2 + xy - xz$ express from here c_2

$2xy + y^2 - yz - x^2 - xy + xz = c_2$

$c_2 = xy + y^2 - yz - x^2 + xz$ **the second first integral**

Relations that define the general integral of system are:

$c_1 = x + y - z$ the first first integral
$c_2 = xy + y^2 - yz - x^2 + xz$ **the second first integral**