Integrals-tasks (V part)

Integrals of some functions that include quadratic equation ax^2+bx+c

First, we study the integrals form: $I_1=\int\frac{dx}{ax^2+bx+c}$ and $I_3=\int\frac{dx}{\sqrt{ax^2+bx+c}}$

Here the quadratic equation ax^2+bx+c is reduced to the canonical form using the formula:

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$$

Of course, we can use and recharge to full squares, who does not like to remember the formula.

Then take the substitute $\left|\begin{matrix}x+\frac{b}{2a}=t\\ dx=dt\end{matrix}\right|$. We get some of the tablet integrals.

$I_1=\int\frac{dx}{ax^2+bx+c}$ can be reduced to $\int\frac{1}{a^2+x^2}dx=\frac{1}{a}arctg\frac{x}{a}+C$, $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$

or $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$

$I_3=\int\frac{dx}{\sqrt{ax^2+bx+c}}$ can be reduced to $\int\frac{1}{\sqrt{a^2-x^2}}dx=\arcsin\frac{x}{a}+C$ or $\int\frac{dx}{\sqrt{x^2\pm a^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$

Example 1. $\int\frac{dx}{x^2-6x+13}=?$

$x^2-6x+13=$ Here are: $a=1,\ b=-6,\ c=13$ and replace that in formula $a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$, so:

$$a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}=1\left(x+\frac{-6}{2\cdot1}\right)^2+\frac{4\cdot1\cdot13-(-6)^2}{4\cdot1}=(x-3)^2+\frac{52-36}{4}=\boxed{(x-3)^2+4}$$

We can amend this to a full square…

$$x^2-6x+13=\underline{x^2-6x+9}-9+13=\boxed{(x-3)^2+4}$$

You work as required by your professor…

We return to the integral:

$$\int \frac{dx}{x^2-6x+13} = \int \frac{dx}{(x-3)^2+4} = \begin{vmatrix} x-3=t \\ dx=dt \end{vmatrix} = \int \frac{dt}{t^2+2^2}$$ (this is $\int \frac{1}{a^2+x^2}dx = \frac{1}{a} arctg \frac{x}{a} + C$ from table of integrals)=

$$\frac{1}{2} arctg \frac{t}{2} + C = \boxed{\frac{1}{2} arctg \frac{x-3}{2} + C}$$

Example 2. $\int \frac{dx}{\sqrt{2x^2-6x+5}} = ?$

$$2x^2-6x+5 = 2(x^2-3x+\frac{5}{2}) = 2(x^2-3x+\frac{9}{4}-\frac{9}{4}+\frac{5}{2}) = 2[(x-\frac{3}{2})^2+\frac{1}{4}]$$

$$\int \frac{dx}{\sqrt{2x^2-6x+5}} = \int \frac{dx}{\sqrt{2[(x-\frac{3}{2})^2+\frac{1}{4}]}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x-\frac{3}{2})^2+\frac{1}{4}}} = \begin{vmatrix} x-\frac{3}{2}=t \\ dx=dt \end{vmatrix} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{t^2+(\frac{1}{2})^2}}$$

This is tablet integral: $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C$

$$\frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{t^2+(\frac{1}{2})^2}} = \frac{1}{\sqrt{2}} \ln\left|t+\sqrt{t^2+(\frac{1}{2})^2}\right| + C = \boxed{\frac{1}{\sqrt{2}} \ln\left|x-\frac{3}{2}+\sqrt{2x^2-6x+5}\right| + C}$$

When we know this two types of integrals, we can learn:

$$\boxed{I_2 = \int \frac{Ax+B}{ax^2+bx+c}dx} \quad \text{and} \quad \boxed{I_4 = \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx}$$

They can work down to the previous two integrals:

$I_2 = \int \frac{Ax+B}{ax^2+bx+c}dx$ is reduced to the integral $I_1 = \int \frac{dx}{ax^2+bx+c}$, while

$I_4 = \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$ is reduced to the integral $I_3 = \int \frac{dx}{\sqrt{ax^2+bx+c}}$.

There are formulas in which only needs to compare and find the values of A,B,a,b and c.

Beware: it may only be used if approved by your professor! We will show you the entire process in case you may not use the formula ...

Formulas are:

$$I_2 = \frac{A}{2a}\ ln\ \left|ax^2+bx+c\right| + (B - \frac{Ab}{2a})\ I_1 + C$$

and

$$I_4 = \frac{A}{a}\sqrt{ax^2+bx+c} + \left(B - \frac{A\cdot b}{2a}\right) I_3 + C$$

Example 3. $\int \frac{x+1}{x^2+x+1}dx = ?$

This is obviously an integral type $I_2 = \int \frac{Ax+B}{ax^2+bx+c}dx$

Comparing we get that: *A=1, B=1, a=1, b=1,c=1*

$$I_2 = \frac{A}{2a}\ln\left|ax^2+bx+c\right| + (B - \frac{Ab}{2a})I_1 + C$$

$$A=1, B=1, a=1, b=1, c=1$$

$$I_2 = \frac{1}{2\cdot 1}\ln\left|1x^2+1x+1\right| + (1 - \frac{1\cdot 1}{2\cdot 1})I_1 + C = \boxed{\frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{2}I_1 + C}$$

Now we have to solve the integral type $I_1 = \int \frac{1}{x^2+x+1}dx$ and his solution to return in the formula

$$I_1 = \int \frac{1}{x^2+x+1}dx = ?$$

$$x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$I_1 = \int \frac{1}{x^2+x+1}dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx = \left| \begin{matrix} x+\frac{1}{2}=t \\ dx = dt \end{matrix} \right| = \int \frac{1}{t^2+(\frac{\sqrt{3}}{2})^2}dx = \frac{1}{\frac{\sqrt{3}}{2}}arctg\frac{t}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}arctg\frac{2(x+\frac{1}{2})}{\sqrt{3}} =$$

$$= \boxed{\frac{2}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}}}$$

Go back:

$$\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{2}I_1 + C = \frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{\not 2}\cdot\frac{\not 2}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}} + C$$

$$\boxed{\frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}} + C}$$

This task would be solved without using formula as follows:

$$\int \frac{x+1}{x^2+x+1}dx = ?$$

$$\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+2}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+1+1}{x^2+x+1}dx = \frac{1}{2}(\int \frac{2x+1}{x^2+x+1}dx + \int \frac{1}{x^2+x+1}dx)$$

Now we solve each of these two integrals, particular.

For first we will use substitution:

$$\int \frac{2x+1}{x^2+x+1}dx = \begin{vmatrix} x^2+x+1=t \\ (2x+1)dx = dt \end{vmatrix} = \int \frac{1}{t}dt = \ln|t| = \ln|x^2+x+1|$$

Here is another integral solution:

$$\int \frac{1}{x^2+x+1}dx = \int \frac{1}{x^2+x+1}dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx = \begin{vmatrix} x+\frac{1}{2}=t \\ dx = dt \end{vmatrix} = \int \frac{1}{t^2+(\frac{\sqrt{3}}{2})^2}dx = \frac{1}{\frac{\sqrt{3}}{2}} arctg \frac{t}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} arctg \frac{2(x+\frac{1}{2})}{\sqrt{3}} =$$

$$= \frac{2}{\sqrt{3}} arctg \frac{2x+1}{\sqrt{3}}$$

Go back in task:

$$\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}(\int \frac{2x+1}{x^2+x+1}dx + \int \frac{1}{x^2+x+1}dx) =$$

$$= \frac{1}{2}(\ln|x^2+x+1| + \frac{2}{\sqrt{3}} arctg \frac{2x+1}{\sqrt{3}}) + C$$

$$= \boxed{\frac{1}{2}\ln|x^2+x+1| + \frac{1}{\sqrt{3}} arctg \frac{2x+1}{\sqrt{3}} + C}$$

Example 4. $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = ?$

first way (with the help of the formula)

$$I_4 = \frac{A}{a}\sqrt{ax^2+bx+c} + \left(B - \frac{A\cdot b}{2a}\right)I_3$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = ?$$

$A = 5, B = 3, a = 1, b = 4, c = 10$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{1}\sqrt{x^2+4x+10} + \left(3 - \frac{5\cdot 4}{2\cdot 1}\right)\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= 5\cdot\sqrt{x^2+4x+10} + (-7)\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= 5\cdot\sqrt{x^2+4x+10} - 7\cdot\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

To solve this integral separately, and will return its solution…

$$\int \frac{1}{\sqrt{x^2+4x+10}}dx =$$

$$x^2+4x+10 = x^2+4x+4-4+10 = (x+2)^2+6$$

$$\int \frac{1}{\sqrt{x^2+4x+10}}dx = \int \frac{1}{\sqrt{(x+2)^2+6}}dx = \begin{vmatrix} x+2=t \\ dx=dt \end{vmatrix} = \int \frac{1}{\sqrt{t^2+6}}dx = here\ we\ use: \boxed{\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C}$$

$$= \ln\left|t+\sqrt{t^2+6}\right| = \ln\left|x+2+\sqrt{x^2+4x+10}\right|$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = 5\cdot\int \frac{x+2}{\sqrt{x^2+4x+10}}dx - 7\cdot\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \boxed{5\cdot\sqrt{x^2+4x+10} - 7\cdot\ln\left|x+2+\sqrt{x^2+4x+10}\right| + C}$$

second way (directly, without using formula)

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = ?$$

As derivate is $(x^2+4x+10)` = 2x+4 = 2(x+2)$ in the numerator must be made that.

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{5(x+\frac{3}{5})}{\sqrt{x^2+4x+10}}dx = 5\cdot\int \frac{x+2-2+\frac{3}{5}}{\sqrt{x^2+4x+10}}dx =$$

$$= 5\cdot(\int \frac{x+2}{\sqrt{x^2+4x+10}}dx + \int \frac{-2+\frac{3}{5}}{\sqrt{x^2+4x+10}}dx)$$

$$= 5\cdot(\int \frac{x+2}{\sqrt{x^2+4x+10}}dx + \int \frac{-\frac{7}{5}}{\sqrt{x^2+4x+10}}dx)$$

$$= 5\cdot\int \frac{x+2}{\sqrt{x^2+4x+10}}dx - 7\cdot\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Now do these two integrals…

$$\int \frac{x+2}{\sqrt{x^2+4x+10}}dx = \left| \begin{array}{l} x^2+4x+10 = t^2 \\ (2x+4)dx = 2tdt \\ \cancel{2}(x+2)dx = \cancel{2}tdt \\ (x+2)dx = tdt \end{array} \right| = \int \frac{\cancel{t}dt}{\cancel{t}} = \int dt = t = \sqrt{x^2+4x+10}$$

$$\int \frac{1}{\sqrt{x^2+4x+10}}dx =$$

$$x^2+4x+10 = x^2+4x+4-4+10 = (x+2)^2+6$$

$$\int \frac{1}{\sqrt{x^2+4x+10}}dx = \int \frac{1}{\sqrt{(x+2)^2+6}}dx = \left| \begin{array}{l} x+2=t \\ dx=dt \end{array} \right| = \int \frac{1}{\sqrt{t^2+6}}dx = we\ use: \boxed{\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C}$$

$$= \ln\left|t+\sqrt{t^2+6}\right| = \ln\left|x+2+\sqrt{x^2+4x+10}\right|$$

Let's go back to the task:

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = 5\cdot\int \frac{x+2}{\sqrt{x^2+4x+10}}dx - 7\cdot\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \boxed{5\cdot\sqrt{x^2+4x+10} - 7\cdot\ln\left|x+2+\sqrt{x^2+4x+10}\right| + C}$$

Example 5. $\int \frac{2x+7}{x^2+x-2}dx = ?$

Would say that this is a integral type $I_2 = \int \frac{Ax+B}{ax^2+bx+c}dx$ and worked to:

$$\int \frac{2x+7}{x^2+x-2}dx = \int \frac{2x+1+6}{x^2+x-2}dx = \int \frac{2x+1}{x^2+x-2}dx + \int \frac{6}{x^2+x-2}dx =$$

Solve these two integrals separately, and then we will replace their solutions ...

$$\int \frac{2x+1}{x^2+x-2}dx = \begin{vmatrix} x^2+x-2=t \\ (2x+1)dx = dt \end{vmatrix} = \int \frac{dt}{t} = \ln|t| = \ln\left|x^2+x-2\right|$$

$$\int \frac{6}{x^2+x-2}dx = 6\int \frac{1}{x^2+x-2}dx =$$

$$x^2+x-2 = x^2+x+\frac{1}{4}-\frac{1}{4}-2 = (x+\frac{1}{2})^2-\frac{9}{4}$$

$$6\int \frac{1}{x^2+x-2}dx = 6\int \frac{1}{(x+\frac{1}{2})^2-\frac{9}{4}}dx = \begin{vmatrix} x+\frac{1}{2}=t \\ dx = dt \end{vmatrix} = 6\int \frac{1}{t^2-\left(\frac{3}{2}\right)^2}dx = we\ use: \boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C}$$

$$= 6\cdot\frac{1}{2\cdot\frac{3}{2}}\ln\left|\frac{t-\frac{3}{2}}{t+\frac{3}{2}}\right| = 2\ln\left|\frac{x+\frac{1}{2}-\frac{3}{2}}{x+\frac{1}{2}+\frac{3}{2}}\right| = 2\ln\left|\frac{x-1}{x+2}\right|$$

then we have:

$$\int \frac{2x+7}{x^2+x-2}dx = \int \frac{2x+1}{x^2+x-2}dx + \int \frac{6}{x^2+x-2}dx =$$

$$= \ln\left|x^2+x-2\right| + 2\ln\left|\frac{x-1}{x+2}\right| + C$$

$$= \ln\left|(x-1)(x+2)\right| + \ln\left|\frac{x-1}{x+2}\right|^2 + C$$

$$= \ln\left|(x-1)\cancel{(x+2)}\cdot\frac{(x-1)^2}{(x+2)^{\cancel{2}}}\right| + C$$

$$\boxed{= \ln\left|\frac{(x-1)^3}{(x+2)}\right| + C}$$

This task we could solve in another way!

Let's see:

$$x^2+x-2=0 \to x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \to x_1=1, x_2=-2$$

$$\int \frac{2x+7}{x^2+x-2}dx=?$$

$$\frac{2x+7}{x^2+x-2}=\frac{2x+7}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}\text{..................................}/\bullet(x-1)(x+2)$$

$$2x+7=A(x+2)+B(x-1)$$

$$2x+7=Ax+2A+Bx-B$$

$$2x+7=x(A+B)+2A-B$$

$$A+B=2$$

$$\underline{2A-B=7}$$

$$3A=9 \to \boxed{A=3} \to \boxed{B=-1}$$

$$\frac{2x+7}{(x-1)(x+2)}=\frac{3}{x-1}+\frac{-1}{x+2}=\frac{3}{x-1}-\frac{1}{x+2}$$

$$\int \frac{2x+7}{(x-1)(x+2)}dx=\int \frac{3}{x-1}dx-\int \frac{1}{x+2}dx=3\ln|x-1|-\ln|x+2|+C$$

$$=\ln|x-1|^3-\ln|x+2|+C$$

$$\boxed{=\ln\left|\frac{(x-1)^3}{x+2}\right|+C}$$

Our advice is therefore to check whether the quadratic equation in the denominator can be solved, and than work as integration of rational functions.

If quadratic equation is not solvable, do it as an integral type $I_2=\int \frac{Ax+B}{ax^2+bx+c}dx$.

You saw that they “dropped” the same solution

After all, you decide yourself what you prefer and how in turn commanded by Professor ...

The following type of integrals is $\boxed{\int \frac{dx}{(mx+n)\sqrt{ax^2+bx+c}}}$

These integrals we do with substitute : $\left| \begin{array}{l} mx+n=\frac{1}{t} \\ mdx=-\frac{1}{t^2}dt \\ dx=-\frac{1}{m\cdot t^2}dt \end{array} \right|$, and then we have $I_3=\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Example 6. $\int \frac{dx}{x\cdot\sqrt{x^2-4x+1}}=?$

First, take the substitute $x=\frac{1}{t}$ and we have I_3.

$$\int \frac{dx}{x\cdot\sqrt{x^2-4x+1}}=\left| \begin{array}{l} x=\frac{1}{t} \\ dx=-\frac{1}{t^2}dt \end{array} \right|=\int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\cdot\sqrt{(\frac{1}{t})^2-4\cdot\frac{1}{t}+1}}=\int \frac{-\frac{1}{t^{\not 2}}dt}{\frac{1}{\not t}\cdot\sqrt{\frac{1}{t^2}-\frac{4}{t}+1}}=\int \frac{-\frac{1}{t}dt}{\sqrt{\frac{1-4t+t^2}{t^2}}}=$$

$$=\int \frac{-\frac{1}{\not t}dt}{\frac{1}{\not t}\sqrt{t^2-4t+1}}=\int \frac{-dt}{\sqrt{t^2-4t+1}}=-\int \frac{dt}{\sqrt{t^2-4t+1}}$$

$$t^2-4t+1=t^2-4t+4-4+1=(t-2)^2-3$$

$$-\int \frac{dt}{\sqrt{t^2-4t+1}}=-\int \frac{dt}{\sqrt{(t-2)^2-3}}=\left| \begin{array}{l} t-2=z \\ dt=dz \end{array} \right|=-\int \frac{dt}{\sqrt{z^2-3}}=use:\int \frac{dx}{\sqrt{x^2\pm a^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C, so$$

$$=-\ln\left|z+\sqrt{z^2-3}\right|=-\ln\left|t-2+\sqrt{t^2-4t+1}\right|+C$$

$then:$

$$=-\ln\left|t-2+\sqrt{t^2-4t+1}\right|+C=\boxed{-\ln\left|\frac{1}{x}-2+\sqrt{(\frac{1}{x})^2-4\cdot\frac{1}{x}+1}\right|+C}$$

The method of undetermined coefficients (method Ostrogradski)

This method can solve the integral type $\boxed{\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx}$ where in the numerator we have polynomial n-th degree.

The procedure of work is as follows:

- **put together an equation**

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx = Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c} + \lambda\cdot\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

Here is $Q_{n-1}(x)$ polynomial (n-1)- st level of the undetermined coefficients

- **Differentiating this equation**
- **Then multiply all by** $\sqrt{ax^2+bx+c}$
- **On both sides we get polynomials of order *n*, and vague coefficients determined by equating coefficients with the same degree of x.**

Since the polynomial $P_n(x)$ in the tasks is usually second degree, equations will be:

$$\int \frac{mx^2+px+r}{\sqrt{ax^2+bx+c}}dx = (Ax+B)\ \sqrt{ax^2+bx+c} + \lambda\cdot\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

But best to see in the example:

$\boxed{\text{Example 7.}}$ $\int \frac{2x^2+3x}{\sqrt{x^2+2x+2}}dx = ?$

Solution:

$$\int \frac{2x^2+3x}{\sqrt{x^2+2x+2}}dx = (Ax+B)\cdot\sqrt{x^2+2x+2} + \lambda\cdot\int \frac{dx}{\sqrt{x^2+2x+2}} \to \textit{differentiating}$$

$$\frac{2x^2+3x}{\sqrt{x^2+2x+2}} = (Ax+B)`\cdot\sqrt{x^2+2x+2} + (\sqrt{x^2+2x+2})`\cdot(Ax+B) + \frac{\lambda}{\sqrt{x^2+2x+2}}$$

$$\frac{2x^2+3x}{\sqrt{x^2+2x+2}} = A\cdot\sqrt{x^2+2x+2} + \frac{1}{2\sqrt{x^2+2x+2}}(x^2+2x+2)`\cdot(Ax+B) + \frac{\lambda}{\sqrt{x^2+2x+2}}$$

$$\frac{2x^2+3x}{\sqrt{x^2+2x+2}} = A\cdot\sqrt{x^2+2x+2} + \frac{1}{2\sqrt{x^2+2x+2}}(2x+2)\cdot(Ax+B) + \frac{\lambda}{\sqrt{x^2+2x+2}}$$

$$\frac{2x^2+3x}{\sqrt{x^2+2x+2}} = A\cdot\sqrt{x^2+2x+2} + \frac{1}{\not{2}\sqrt{x^2+2x+2}}\not{2}(x+1)\cdot(Ax+B) + \frac{\lambda}{\sqrt{x^2+2x+2}}$$

$$\frac{2x^2+3x}{\sqrt{x^2+2x+2}} = A\cdot\sqrt{x^2+2x+2} + \frac{(x+1)\cdot(Ax+B)}{\sqrt{x^2+2x+2}} + \frac{\lambda}{\sqrt{x^2+2x+2}} \to \ldots\ldots\ldots\ldots\ldots /\bullet\sqrt{x^2+2x+2}$$

$$2x^2+3x = A(x^2+2x+2) + (x+1)\cdot(Ax+B) + \lambda$$

Now compare coefficients:

$$2x^2+3x = A(x^2+2x+2) + (x+1)\cdot(Ax+B) + \lambda$$

$$2x^2+3x = \underline{Ax^2} + \overline{2Ax} + 2A + \underline{Ax^2} + \overline{Bx} + \overline{Ax} + B + \lambda$$

$$2x^2+3x = 2Ax^2 + x(3A+B) + 2A + B + \lambda \to \textit{compare}$$

$$2A = 2 \to \boxed{A=1}$$

$$3A+B=3$$

$$\underline{2A+B+\lambda=0}$$

$$3A+B=3 \to 3\cdot1+B=3 \to \boxed{B=0}$$

$$2A+B+\lambda=0 \to 2+0+\lambda=0 \to \boxed{\lambda=-2}$$

Return to the starting equation:

$$\int \frac{2x^2+3x}{\sqrt{x^2+2x+2}}dx = (Ax+B)\cdot\sqrt{x^2+2x+2} + \lambda\cdot\int\frac{dx}{\sqrt{x^2+2x+2}}$$

$$= (1x+0)\cdot\sqrt{x^2+2x+2} - 2\cdot\int\frac{dx}{\sqrt{x^2+2x+2}}$$

$$= x\cdot\sqrt{x^2+2x+2} - 2\cdot\int\frac{dx}{\sqrt{x^2+2x+2}}$$

To solve this integral:

$$\int\frac{dx}{\sqrt{x^2+2x+2}} = \int\frac{dx}{\sqrt{x^2+2x+1+1}} = \int\frac{dx}{\sqrt{(x+1)^2+1}} = \begin{vmatrix} x+1=t \\ dx=dt \end{vmatrix} = \int\frac{dx}{\sqrt{t^2+1}} =$$

$$we\ use: \int\frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C$$

$$= \ln\left|t+\sqrt{t^2+1}\right| + C = \boxed{\ln\left|x+1+\sqrt{x^2+2x+2}\right| + C}$$

Finally, the solution will be:

$$\int \frac{2x^2+3x}{\sqrt{x^2+2x+2}}dx = x\cdot\sqrt{x^2+2x+2} - 2\cdot\int\frac{dx}{\sqrt{x^2+2x+2}}$$

$$= \boxed{x\cdot\sqrt{x^2+2x+2} - 2\cdot\ln\left|x+1+\sqrt{x^2+2x+2}\right| + C}$$

Example 8.

$$\int\sqrt{a^2-x^2}\,dx = ?$$

If you remember, we solve this integral with partial integration .Then we say that it can be solved in several ways. Here's how to work out with the method of Ostrogradski.

Of course, again a little rationalization…

$$\frac{\sqrt{a^2-x^2}}{1}\cdot\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}=\frac{a^2-x^2}{\sqrt{a^2-x^2}}$$

$$\int\sqrt{a^2-x^2}dx=\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}dx$$

Now this is the form that we need …

$$\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}dx=(Ax+B)\cdot\sqrt{a^2-x^2}+\lambda\cdot\int\frac{dx}{\sqrt{a^2-x^2}}\to differentiating$$

$$\frac{a^2-x^2}{\sqrt{a^2-x^2}}=A\sqrt{a^2-x^2}+\frac{-\not{2}x}{\not{2}\sqrt{a^2-x^2}}(Ax+B)+\frac{\lambda}{\sqrt{a^2-x^2}}$$

$$\frac{a^2-x^2}{\sqrt{a^2-x^2}}=A\sqrt{a^2-x^2}+\frac{-x}{\sqrt{a^2-x^2}}(Ax+B)+\frac{\lambda}{\sqrt{a^2-x^2}}\text{..}/\bullet\sqrt{a^2-x^2}$$

$$a^2-x^2=A(a^2-x^2)-x(Ax+B)+\lambda$$

$$a^2-x^2=Aa^2-Ax^2-Ax^2-Bx+\lambda$$

$$a^2-x^2=-2Ax^2-Bx+Aa^2+\lambda$$

$compare$

$$-2A=-1$$

$$-B=0\to\boxed{B=0}$$

$$\underline{Aa^2+\lambda=a^2}$$

$system:$

$$\boxed{A=\frac{1}{2}}\to\frac{1}{2}a^2+\lambda=a^2\to\boxed{\lambda=\frac{1}{2}a^2}$$

Go back, and we have solution:

$$\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}dx=(Ax+B)\cdot\sqrt{a^2-x^2}+\lambda\cdot\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}dx=\frac{1}{2}x\cdot\sqrt{a^2-x^2}+\frac{1}{2}a^2\cdot\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$\boxed{\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}dx=\frac{1}{2}x\cdot\sqrt{a^2-x^2}+\frac{1}{2}a^2\cdot\arcsin\frac{x}{a}+C}$$