Integrals-tasks (IV part)- Integration of rational functions

Rational function is $\frac{P(x)}{Q(x)}$. It may be **right and improper.**

Right rational function is one in which the maximum degree of polynomial P (x) is less than the maximum level polynomial Q (x). (For example: $\frac{x}{x^2-3}, \frac{x^2-3x+12}{x^3+5x^2-2x+1}, \frac{4}{x^5+5x^2-22x+31}$ …etc.)

Improper rational function is the one at which the maximum degree of P (x) is greater or equal to the maximum degree of Q (x)(For example: $\frac{x^2+2x-7}{x^2-13}, \frac{x^2-3x+12}{2x+1}, \frac{x^4+75x^2-14}{x+3}$ …etc.)

In the case that is given an improper rational function , **we** divide the two polynomials, get the solution plus right rational function.

Integration of rational functions we make by:

Denominator reduced to factors by using:

- derive a common front brackets
- difference of squares: $a^2 - b^2 = (a-b)(a+b)$
- $ax^2 + bx + c = a(x-x_1)(x-x_2)$ If we are given quadratic equation
- $a^3-b^3 = (a-b)(a^2+ab+b^2)$ or $a^3+b^3 = (a+b)(a^2-ab+b^2)$
- We use Bézout theorem…

Further given <u>right rational function</u> decomposes as follows: (examples)

$\frac{P(x)}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5}$ if in denominator they are all linear with no degree, every one goes with the letter: A, B, C ...

$\frac{P(x)}{(x-1)^3(x+7)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x+7)}$ if in denominator we have the linear members, but with degree, divide until they reach the highest level.

$\frac{P(x)}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$ if in denominator is irresolvable polynomial , for it must take $Bx+C$

Here's an example in which all three things occur:

$$\frac{P(x)}{(x-7)\cdot(x^2+5)^2\cdot x^3} = \frac{A}{x-7} + \frac{Bx+C}{x^2+5} + \frac{Dx+E}{(x^2+5)^2} + \frac{F}{x} + \frac{G}{x^2} + \frac{H}{x^3}$$

EXSAMPLES

$\boxed{1.}$ $\int \frac{x-3}{x^3-x}dx = ?$

First to denominator reduced to factors!

$x^3 - x = x(x^2-1) = x(x-1)(x+1)$

Then, we have: $\int \frac{x-3}{x(x-1)(x+1)}dx = ?$

Draw a rational function:

$\frac{x-3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$ all multiply by $x(x-1)(x+1)$

$x-3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

$x-3 = A(x^2-1) + Bx^2 + Bx + Cx^2 - Cx$

$x-3 = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$

$\underline{x-3 = x^2(A+B+C) + x(B-C) - A}$

compared members with x^2, then with x and free members

$A+B+C=0$

$B-C=1$

$-A=-3$

Solve this system of equations:

$A+B+C=0$

$B-C=1$

$\underline{-A=-3}$

$\boxed{A=3} \rightarrow A+B+C=0 \rightarrow 3+B+C=0 \rightarrow B+C=-3$

$\boxed{\begin{matrix} B-C=1 \\ B+C=-3 \end{matrix}} \rightarrow 2B=-2 \rightarrow \boxed{B=-1} \rightarrow \boxed{C=-2}$

Back with solutions:

$\frac{x-3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

$\frac{x-3}{x(x-1)(x+1)} = \frac{3}{x} + \frac{-1}{x-1} + \frac{-2}{x+1} = \frac{3}{x} - \frac{1}{x-1} - \frac{2}{x+1}$

Now go back to solve the given integral, as we have it apart in three small integrals, which are mostly tablet…

$$\int \frac{x-3}{x(x-1)(x+1)}dx = \int \frac{3}{x}dx - \int \frac{1}{x-1}dx - \int \frac{2}{x+1}dx$$

$$= \boxed{3\ln|x| - \ln|x-1| - 2\ln|x+1| + C}$$

Maybe your teacher will require that the "pack" solution using the rules of logarithms. To remind:

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln(x \cdot y) = \ln x + \ln y$
4. $\ln \frac{x}{y} = \ln x - \ln y$
5. $\ln x^n = n \ln x$
6. $e^{\ln x} = x$

Our solution will be:

$$3\ln|x| - \ln|x-1| - 2\ln|x+1| + C = \ln|x|^3 - (\ln|x-1| + \ln|x+1|^2) + C = \ln|x|^3 - \ln|x-1|\cdot|x+1|^2 + C =$$

$$= \boxed{\ln\left|\frac{x^3}{(x-1)(x+1)^2}\right| + C}$$

$\boxed{2.}$ $\int \frac{x+2}{x^3-2x^2}dx = ?$

$$\frac{x+2}{x^3-2x^2} = \frac{x+2}{x^2(x-2)}$$

$$\frac{x+2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$ all multiply by $x^2(x-2)$

$x+2 = Ax(x-2) + B(x-2) + Cx^2$

$x+2 = Ax^2 - 2Ax + Bx - 2B + Cx^2$

$1 \cdot x + 2 = x^2(A+C) + x(-2A+B) - 2B$ compare

$A + C = 0$

$-2A + B = 1$

$\underline{-2B = 2}$

the third equation immediately gives the value for B

$\boxed{B=-1} \to -2A + B = 1 \to -2A - 1 = 1 \to \boxed{A=-1} \to \boxed{C=1}$

$$\frac{x+2}{x^2(x-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$$

$$\frac{x+2}{x^2(x-2)}=\frac{-1}{x}+\frac{-1}{x^2}+\frac{1}{x-2}$$

$$\int\frac{x+2}{x^2(x-2)}dx=\int\frac{-1}{x}dx+\int\frac{-1}{x^2}dx+\int\frac{1}{x-2}dx= -\int\frac{1}{x}dx-\int x^{-2}dx+\int\frac{1}{x-2}dx=-\ln|x|-\frac{x^{-2+1}}{-2+1}+\ln|x-2|+C$$

sulition is :

$$\ln|x-2|-\ln|x|+\frac{1}{x}+C=\boxed{\ln\left|\frac{x-2}{x}\right|+\frac{1}{x}+C}$$

$\boxed{3.}$ $\int\frac{x^3+x^2-16x+16}{x^2-4x+3}dx=?$

This is an improper rational function (degree of numerator is greater than the degree of the denominator), so we split these two polynomial.

$$\frac{x^3+x^2-16x+16}{x^2-4x+3}=$$

$$(x^3+x^2-16x+16):(x^2-4x+3)=x+5$$

$$\underline{\pm x^3\mp 4x^2\pm 3x}$$

$$+5x^2-19x+16$$

$$\underline{\pm 5x^2\mp 20x\pm 15}$$

$$\boxed{x+1}\rightarrow rest$$

$$\frac{x^3+x^2-16x+16}{x^2-4x+3}=x+5+\frac{x+1}{x^2-4x+3}$$

We got a right rational function that continues to divide:

$$\frac{x+1}{x^2-4x+3}=$$

$$x^2-4x+3=0 \to x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \to x_{1,2}=\frac{4\pm 2}{2} \to x_1=3, x_2=1$$

$$x^2-4x+3=(x-1)(x-3)$$

$$\frac{x+1}{(x-1)(x-3)}=\frac{A}{x-1}+\frac{B}{x-3}$$

$$x+1=A(x-3)+B(x-1)$$

$$x+1=Ax-3A+Bx-B$$

$$x+1=x(A+B)-3A-B$$

$$A+B=1$$

$$\underline{-3A-B=1}$$

$$-2A=2 \to \boxed{A=-1} \to \boxed{B=2}$$

$$\frac{x+1}{(x-1)(x-3)}=\frac{A}{x-1}+\frac{B}{x-3}=\frac{-1}{x-1}+\frac{2}{x-3}$$

Now solve the whole integral:

$$\frac{x^3+x^2-16x+16}{x^2-4x+3}=x+5+\frac{x+1}{x^2-4x+3}=x+5+\frac{-1}{x-1}+\frac{2}{x-3}$$

$$\int\frac{x^3+x^2-16x+16}{x^2-4x+3}dx=\int(x+5+\frac{-1}{x-1}+\frac{2}{x-3})dx=\frac{x^2}{2}+5x-\ln|x-1|+2\ln|x-3|+C$$

$$=\boxed{\frac{x^2}{2}+5x+\ln\frac{(x-3)^2}{|x-1|}+C}$$

$\boxed{4.}$ $$\int\frac{xdx}{x^3-3x+2}=?$$

$$\frac{x}{x^3-3x+2}=$$

First, to function in the denominator reduced to its factors

$$x^3-3x+2=0 \qquad \text{idea is}: -3x=-x-2x$$

$$x^3-x-2x+2=0$$

$$x(x^2-1)-2(x-1)=0$$

$$x\boxed{(x-1)}(x+1)-2\boxed{(x-1)}=0$$

$$(x-1)[x(x+1)-2]=0$$

$$(x-1)(x^2+x-2)=0 \to x-1=0 \vee x^2+x-2=0 \to x_1=1, x_2=1, x_3=-2$$

$$x^3-3x+2=(x-1)(x-1)(x+2)=(x-1)^2(x+2)$$

$$\frac{x}{x^3-3x+2}=\frac{x}{(x-1)^2(x+2)}$$

$$\frac{x}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2}\text{..................................}/\bullet(x-1)^2(x+2)$$

$$x=A(x-1)(x+2)+B(x+2)+C(x-1)^2$$

$$x=A(x^2+x-2)+Bx+2B+C(x^2-2x+1)$$

$$x=Ax^2+Ax-2A+Bx+2B+Cx^2-2Cx+C$$

$$\underline{1\cdot x=x^2(A+C)+x(A+B-2C)-2A+2B+C}$$ now compare

$$A+C=0$$

$$A+B-2C=1$$

$$\underline{-2A+2B+C=0}$$

$$C=-A$$

$$A+B+2A=1$$

$$\underline{-2A+2B-A=0}$$

$$3A+B=1$$

$$\underline{-3A+2B=0}$$

$$3B=1\rightarrow\boxed{B=\frac{1}{3}}\rightarrow 3A=\frac{2}{3}\rightarrow\boxed{A=\frac{2}{9}}\rightarrow\boxed{C=-\frac{2}{9}}$$

$$\frac{x}{(x-1)^2(x+2)}=\frac{\frac{2}{9}}{x-1}+\frac{\frac{1}{3}}{(x-1)^2}+\frac{-\frac{2}{9}}{x+2}$$

to solve the integral:

$$\int\frac{x}{(x-1)^2(x+2)}dx=\int\frac{\frac{2}{9}}{x-1}dx+\int\frac{\frac{1}{3}}{(x-1)^2}dx+\int\frac{-\frac{2}{9}}{x+2}dx=$$

$$=\frac{2}{9}\int\frac{1}{x-1}dx+\frac{1}{3}\int(x-1)^{-2}dx-\frac{2}{9}\int\frac{1}{x+2}dx$$

$$=\boxed{\frac{2}{9}\ln\left|x-1\right|-\frac{1}{3}\cdot\frac{1}{x-1}-\frac{2}{9}\ln\left|x+2\right|+C}$$

$\boxed{5.}$ $\int \frac{xdx}{x^3 - x^2 + x - 1} = ?$

$$\frac{x}{x^3 - x^2 + x - 1} =$$

$$x^3 - x^2 + x - 1 = x^2(x-1) + 1(x-1) = (x-1)(x^2+1)$$

$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \rightarrow$ x^2+1 is indecomposable in R

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \text{..........................} / \bullet (x-1)(x^2+1)$$

$$x = A(x^2+1) + (Bx+C)(x-1)$$

$$x = Ax^2 + A + Bx^2 - Bx + Cx - C$$

$$\underline{x = x^2(A+B) + x(-B+C) + A - C}$$

$$A + B = 0$$

$$-B + C = 1$$

$$\underline{A - C = 0} \rightarrow A = C$$

$$A + B = 0$$

$$\underline{A - B = 1}$$

$$2A = 1 \rightarrow \boxed{A = \frac{1}{2}} \rightarrow \boxed{C = \frac{1}{2}} \rightarrow \boxed{B = -\frac{1}{2}}$$

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} = \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{x-1}{x^2+1} = \frac{1}{2}(\frac{1}{x-1} - \frac{x-1}{x^2+1})$$

go back to solve integral:

$$\int \frac{x}{(x-1)(x^2+1)} dx = \int \frac{1}{2}(\frac{1}{x-1} - \frac{x-1}{x^2+1}) dx =$$

$$= \frac{1}{2}\left(\int \frac{1}{x-1} dx - \int \frac{x}{x^2-1} dx + \int \frac{1}{x^2+1} dx \right)$$

$$= \boxed{\frac{1}{2}\left(\ln|x-1| - \frac{1}{2}\ln|x^2-1| + arctgx \right) + C}$$

$\boxed{6.}$ $\int \frac{4}{x^4+1}dx = ?$

This is a serious task!

$$\frac{4}{x^4+1} =$$

Here's the problem: How to decompose denominator to the factors?

The trick is to add and subtract $2x^2$...

$$x^4+1 = \underline{x^4+2x^2+1} - 2x^2 = (x^2+1)^2 - 2x^2 = (x^2+1)^2 - (\sqrt{2}x)^2 = (x^2+1-\sqrt{2}x)\cdot(x^2+1+\sqrt{2}x)$$

$$x^4+1 = (x^2-\sqrt{2}x+1)\cdot(x^2+\sqrt{2}x+1)$$

$$\frac{4}{x^4+1} = \frac{4}{(x^2-\sqrt{2}x+1)\cdot(x^2+\sqrt{2}x+1)}$$

$$\frac{4}{(x^2-\sqrt{2}x+1)\cdot(x^2+\sqrt{2}x+1)} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1} \ldots\ldots\ldots\ldots\ldots\ldots / \bullet (x^2-\sqrt{2}x+1)\cdot(x^2+\sqrt{2}x+1)$$

$$4 = (Ax+B)(x^2+\sqrt{2}x+1) + (Cx+D)(x^2-\sqrt{2}x+1)$$

$$4 = Ax^3 + A\sqrt{2}x^2 + Ax + Bx^2 + B\sqrt{2}x + B + Cx^3 - C\sqrt{2}x^2 + Cx + Dx^2 - D\sqrt{2}x + D$$

$$4 = x^3(A+C) + x^2(A\sqrt{2}+B-C\sqrt{2}+D) + x(A+B\sqrt{2}+C-D\sqrt{2}) + B + D$$

compare:

$A+C=0$

$A\sqrt{2}+B-C\sqrt{2}+D=0$

$A+B\sqrt{2}+C-D\sqrt{2}=0 \rightarrow \underline{|A+C=0 \rightarrow B\sqrt{2}-D\sqrt{2}=0 \rightarrow \sqrt{2}(B-D)=0 \rightarrow B-D=0}$

$\underline{B+D=4}$

$B-D=0$

$\underline{B+D=4}$

$\boxed{B=2} \wedge \boxed{D=2}$

$A+C=0$

$\underline{\sqrt{2}(A-C)=-4}$

$\boxed{A=-\sqrt{2}} \wedge \boxed{C=\sqrt{2}}$

We got:

$$\frac{4}{(x^2-\sqrt{2}x+1)\cdot(x^2+\sqrt{2}x+1)} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1}$$

$$\frac{4}{(x^2-\sqrt{2}x+1)\cdot(x^2+\sqrt{2}x+1)} = \frac{-\sqrt{2}x+2}{x^2-\sqrt{2}x+1} + \frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1}$$

We have therefore to solve:

$$\int \frac{4}{(x^2-\sqrt{2}x+1)\cdot(x^2+\sqrt{2}x+1)}dx = \int \frac{-\sqrt{2}x+2}{x^2-\sqrt{2}x+1}dx + \int \frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1}dx$$

These are integral type $I_2=\int \frac{Ax+B}{ax^2+bx+c}dx$ to be solved over $I_1=\int \frac{dx}{ax^2+bx+c}$ and formula:

$$I_2=\frac{A}{2a}\ln\left|ax^2+bx+c\right|+\left(B-\frac{Ab}{2a}\right)I_1+C$$

Solving procedure is explained in one of the files integrals – tasks…

Here is the final solution and you check it out…

$$\boxed{\frac{1}{\sqrt{2}}\cdot\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\sqrt{2}\cdot arctg\frac{x\sqrt{2}}{1-x^2}+C}$$

7.

$$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$ **Prove!**

Proof:

$$\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}$$

$$\frac{1}{(x-a)(x+a)}=\frac{A}{x-a}+\frac{B}{x+a}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots / \bullet(x-a)(x+a)$$

$$1=A(x+a)+B(x-a)$$

$$1=Ax+Aa+Bx-Ba$$

$$1=x(A+B)+Aa-Ba$$

$$compare:$$

$$A+B=0$$

$$\underline{a(A-B)=1}$$

$$A+B=0$$

$$\underline{A-B=\frac{1}{a}}$$

$$2A=\frac{1}{a}\rightarrow\boxed{A=\frac{1}{2a}}\wedge\boxed{B=-\frac{1}{2a}}$$

$$\frac{1}{(x-a)(x+a)}=\frac{A}{x-a}+\frac{B}{x+a}$$

$$\frac{1}{(x-a)(x+a)}=\frac{\frac{1}{2a}}{x-a}+\frac{-\frac{1}{2a}}{x+a}=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$$

$$\int \frac{1}{(x-a)(x+a)}\,dx = \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}\left(\int \frac{1}{x-a}\,dx - \int \frac{1}{x+a}\,dx\right)$$

$$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C$$

$$= \boxed{\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C}$$