MULTIPLE INTEGRALS - TASKS (IV PART)

CALCULATING THE VOLUME USING DOUBLE INTEGRAL

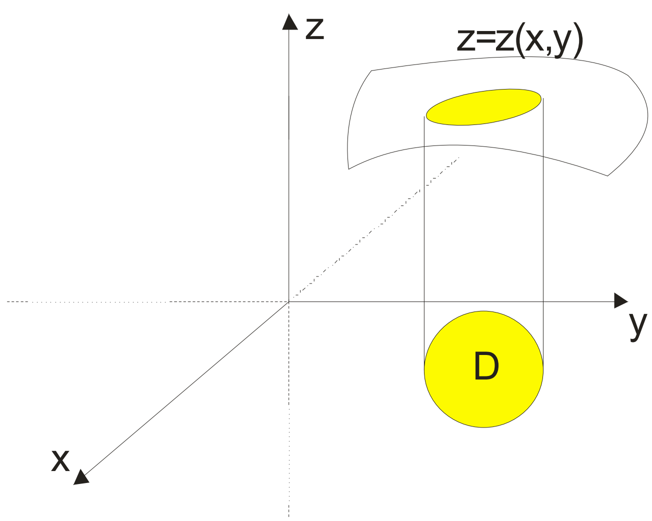
Before you start with the study of this file , be sure to see the file " Some areas in R^3".

In most of the tasks here is necessary to draw a picture in space, and then when you find the intersection, go down problem in the plane to determine limits.

To remind ourselves of the theoretical part:

Cylinder, which limits on top is the continuous area defined by the equation z = z (x, y) , on bottom plane z = 0, and around cylindrical area , which cuts a plane xOy over area D, has a volume:

$$\mathbf{V} = \iint_D z(x,y)dxdy$$



Thus, the double integral calculates the volume of the **geometric** body below the area within certain limits.

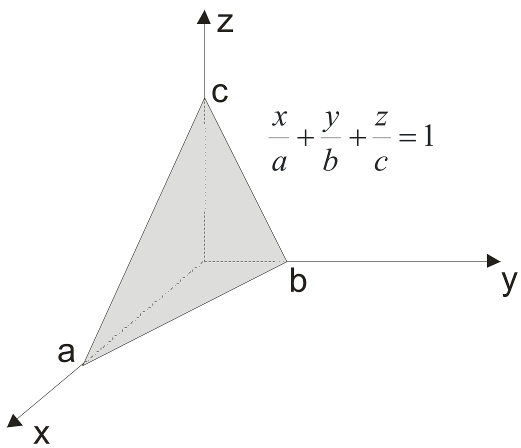
Here are some examples:

Example 1.

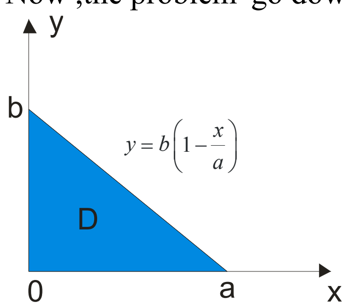
Find a volume limited with the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and $x=0$, $y=0$, $z=0$

Solution:

First, we must draw picture in space:



Now ,the problem 'go down' in xOy plane (ie z = 0) and get



It is clear that x goes from 0 to a.

Determine the line through a and b, since y first “attack” on the x = 0, and then on this line:

$$\frac{x}{a}+\frac{y}{b}=1 \rightarrow \frac{y}{b}=1-\frac{x}{a} \rightarrow y=b\left(1-\frac{x}{a}\right) \rightarrow \boxed{0 \le y \le b\left(1-\frac{x}{a}\right)}$$

We get: $D:\begin{cases} 0 \le x \le a \\ 0 \le y \le b\left(1-\frac{x}{a}\right) \end{cases}$

Now just calculate volume using the formula:

$$V=\iint_D z(x,y)dxdy=\int_0^a dx\int_0^{b(1-\frac{x}{a})} c(1-\frac{x}{a}-\frac{y}{b})dy$$

Solve first:

$$\int c(1-\frac{x}{a}-\frac{y}{b})dy=c(y-\frac{x}{a}y-\frac{y^2}{2b})$$

$$c(y-\frac{x}{a}y-\frac{y^2}{2b})\Bigg|_0^{b(1-\frac{x}{a})}=c\left(b(1-\frac{x}{a})-\frac{x}{a}b(1-\frac{x}{a})-\frac{[b(1-\frac{x}{a})]^2}{2b}\right)=$$

$$c\left(b-\frac{bx}{a}-\frac{bx}{a}+\frac{bx^2}{a^2}-\frac{b^2(1-\frac{2x}{a}+\frac{x^2}{a^2})}{2b}\right)=c\left(b-\frac{2bx}{a}+\frac{bx^2}{a^2}-\frac{b}{2}+\frac{bx}{a}-\frac{bx^2}{2a^2}\right)=$$

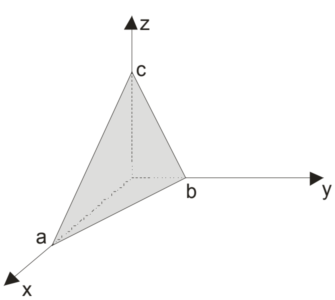
$$c\left(\frac{b}{2}-\frac{2bx}{2a}+\frac{bx^2}{2a^2}\right)=\frac{cb}{2}\left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right)$$

Let us return to integral:

$$V=\iint_D z(x,y)dxdy=\int_0^a \frac{cb}{2}\left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right)dx=$$

$$=\frac{cb}{2}\left(x-\frac{2x^2}{2a}+\frac{x^3}{3a^2}\right)\Bigg|_0^a=\frac{cb}{2}\left(a-\frac{a^2}{a}+\frac{a^3}{3a^2}\right)=\frac{cb}{2}\cdot\frac{a}{3}=\boxed{\frac{abc}{6}}$$

Volume that we have is actually the volume of pyramid!
Let's look at picture again:



Of course, this is much easier to calculate the volume over conventional formula (from high school and even primary).

If we take that the base is triangle ABO, its area is $B=\frac{ab}{2}$, The height of the pyramid is obviously c, so we have:

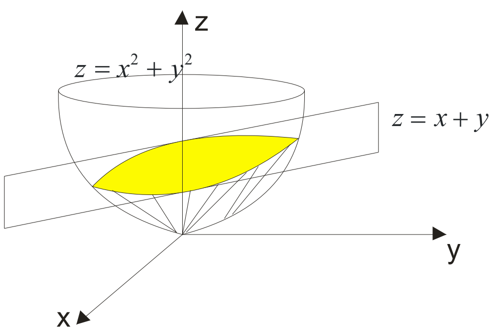
$$V=\frac{1}{3}BH=\frac{1}{3}\frac{ab}{2}\cdot c=\boxed{\frac{abc}{6}}$$

Example 2.

Calculate the volume of the **geometric** body limited with $z = x^2 + y^2$ and $z = x + y$

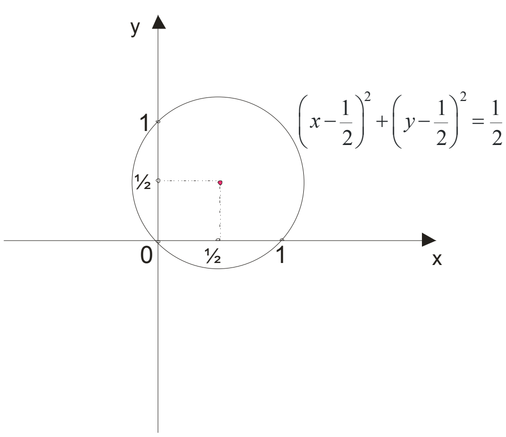
Solution:

Here is a paraboloid $z = x^2 + y^2$ and plane $z = x + y$ which intersects it.



Search volume is the volume inside the paraboloid, that on the top has a plane $z = x + y$

$$x^2 + y^2 = x + y$$
$$x^2 - x + y^2 - y = 0$$
$$x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{4} + \frac{1}{4}$$
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$



We introduce polar coordinates:

$$\left.\begin{aligned} x &= r\cos\varphi + \frac{1}{2} \\ y &= r\sin\varphi + \frac{1}{2} \end{aligned}\right\} \rightarrow |J| = r$$

$$\left(r\cos\varphi+\frac{1}{2}-\frac{1}{2}\right)^2+\left(r\sin\varphi+\frac{1}{2}-\frac{1}{2}\right)^2=\frac{1}{2}$$

$$r^2=\frac{1}{2}\to r=\frac{1}{\sqrt{2}}\to \boxed{0\le r\le\frac{1}{\sqrt{2}}}$$

Let's look at the picture in the plane once again ….

Angle touring the whole circle, so $0\le\varphi\le 2\pi$

$$V=\iint\limits_D\left(z_1(x,y)-z_2(x,y)\right)dxdy$$

$$\left(z_1(x,y)-z_2(x,y)\right)=\left(x+y-(x^2+y^2)\right)=-\left(x^2+y^2-x-y\right)=-\left(x^2-x+\frac{1}{4}+y^2-y+\frac{1}{4}-\frac{1}{2}\right)=$$

$$=-\left((x-\frac{1}{2})^2+(y-\frac{1}{2})^2-\frac{1}{2}\right)=\frac{1}{2}-\left((x-\frac{1}{2})^2+(y-\frac{1}{2})^2\right)=\boxed{\frac{1}{2}-r^2}$$

$$V=\iint\limits_D\left(z_1(x,y)-z_2(x,y)\right)dxdy=\int_0^{2\pi}d\varphi\int_0^{\frac{1}{\sqrt{2}}}\left(\frac{1}{2}-r^2\right)rdr=\int_0^{2\pi}d\varphi\int_0^{\frac{1}{\sqrt{2}}}\left(\frac{r}{2}-r^3\right)dr$$

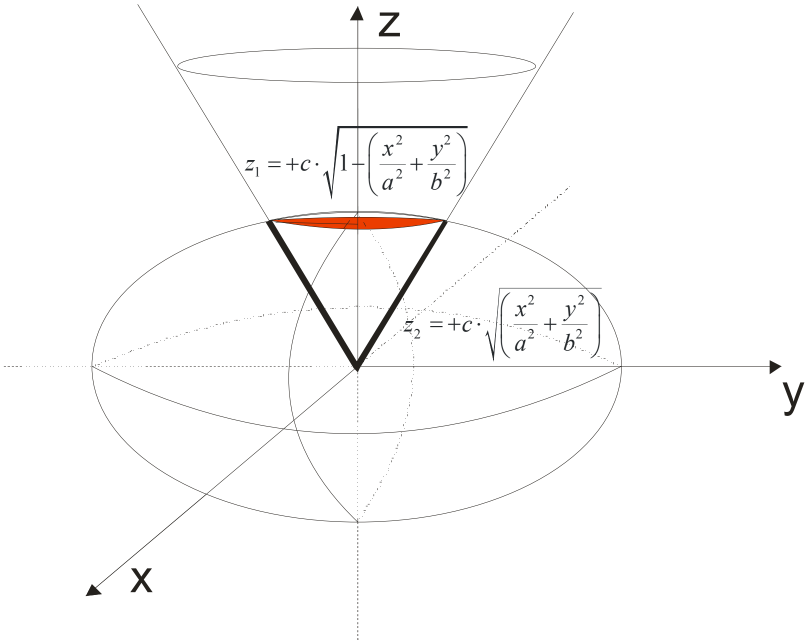
$$=2\pi\cdot\left(\frac{r^2}{4}-\frac{r^4}{4}\right)\Bigg|_0^{\frac{1}{\sqrt{2}}}=2\pi\cdot\left(\frac{\left(\frac{1}{\sqrt{2}}\right)^2}{4}-\frac{\left(\frac{1}{\sqrt{2}}\right)^4}{4}\right)=2\pi\cdot\left(\frac{1}{8}-\frac{1}{16}\right)=2\pi\cdot\frac{1}{16}=\boxed{\frac{\pi}{8}}$$

Example 3.

Calculate the volume limited with $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ if $\left(z>0,a>0,b>0,c>0\right)$

Solution:

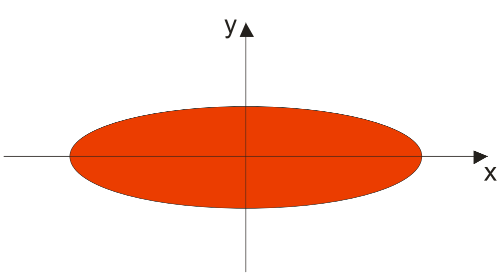
This is about the ellipsoid and the cone. Let's look at the picture:



Search volume is between the two bodies. From above the ellipsoid and the bottom cone!

$$V = \iint_D \left(z_1(x,y) - z_2(x,y) \right) dxdy$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$$



We introduce elliptic coordinates:

$$\left.\begin{aligned} x &= ar\cos\varphi \\ y &= br\sin\varphi \end{aligned}\right\} \to |J| = abr$$

Let's look at the picture in the plane *z = 0 (ellipse)*

It is obvious that the angle taken full circle : $0 \le \varphi \le 2\pi$

$$\left(r\cos\varphi\right)^2+\left(r\sin\varphi\right)^2=\frac{1}{2}$$

$$r^2=\frac{1}{2}\to r=\frac{1}{\sqrt{2}}\to \boxed{0\le r\le\frac{1}{\sqrt{2}}}$$

Before we start the calculation of volume we have to express z in both equations:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{z^2}{c^2}=1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$$

$$z^2=c^2\cdot\left[1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\right]$$

$$z=\pm\sqrt{c^2\cdot\left[1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\right]}$$

$$\boxed{z_1=+c\cdot\sqrt{1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

$$z^2=c^2\cdot\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$$

$$z=\pm\sqrt{c^2\cdot\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}$$

$$\boxed{z_2=+c\cdot\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}}$$

Now here we put the elliptical coordinates:

$$z_1=+c\cdot\sqrt{1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}$$

$$\boxed{z_1=+c\cdot\sqrt{1-r^2}}$$

$$z_2=+c\cdot\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}$$

$$\boxed{z_2=+c\cdot r}$$

Now:

$$V=\iint_D\left(z_1(x,y)-z_2(x,y)\right)dxdy=\int_0^{2\pi}d\varphi\int_0^{\frac{1}{\sqrt{2}}}\left(c\sqrt{1-r^2}-cr\right)abrdr=abc\int_0^{2\pi}d\varphi\int_0^{\frac{1}{\sqrt{2}}}\left(r\sqrt{1-r^2}-r^2\right)dr$$

Now this integral is not difficult to solve: In first we have replacement $\begin{vmatrix}1-r^2=t^2\\-2rdr=2tdt\\rdr=-tdt\end{vmatrix}$, and second is in table of intergals.

We get the volume: $\boxed{V=\frac{abc\cdot\pi}{3}\left(2-\sqrt{2}\right)}$

Example 4.

Calculate the volume limited with $z = x^2 + y^2$, $x^2 + y^2 = x$, $x^2 + y^2 = 2x$, $z = 0$.

Solution:

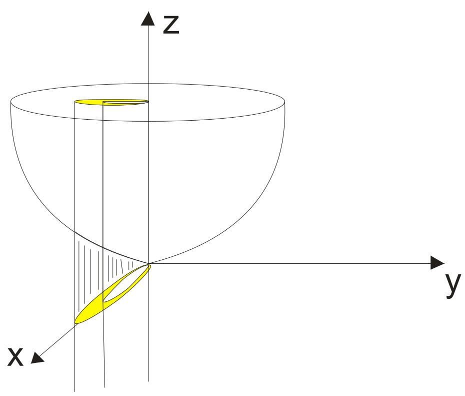
Here is a paraboloid which cuts out two cones ….

$$x^2 + y^2 = x$$
$$x^2 - x + y^2 = 0$$
$$x^2 - x + \frac{1}{4} - \frac{1}{4} + y^2 = 0$$
$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

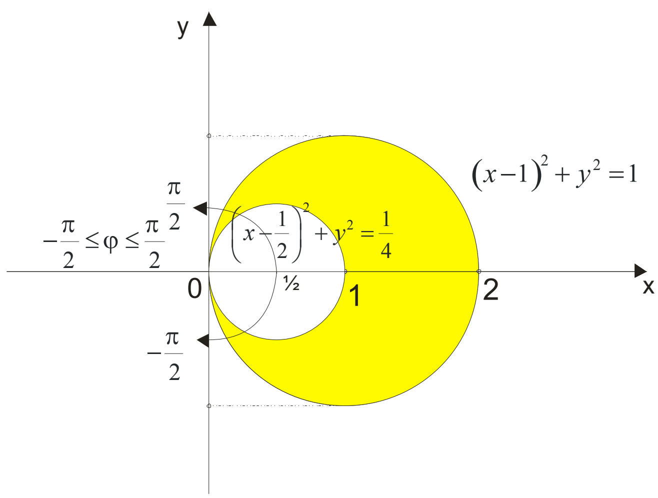
and

$$x^2 + y^2 = 2x$$
$$x^2 - 2x + y^2 = 0$$
$$x^2 - 2x + 1 + y^2 = 1$$
$$(x-1)^2 + y^2 = 1$$

Let's look at the picture :



In $z = 0$ is:



Now we take polar coordinates and determines the boundaries:

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \rightarrow |J| = r$$

$$\begin{aligned} &x^2 + y^2 = x && x^2 + y^2 = 2x \\ &r^2 = r\cos\varphi \quad \text{and} && r^2 = 2r\cos\varphi \quad \text{so:} \quad \boxed{\cos\varphi \le r \le 2\cos\varphi} \\ &r = \cos\varphi && r = 2\cos\varphi \end{aligned}$$

Angle is at the first and fourth quadrants (see picture)

$$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$$

$$V = \iint_D z(x,y)\,dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{\cos\varphi}^{2\cos\varphi} r^2 \cdot r dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Bigg|_{c\,\text{os}\,\varphi}^{2c\,\text{os}\,\varphi} d\varphi = \frac{15}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi d\varphi$$

Since there is symmetry with respect to the x-axis, ie the two parts of equal volume are, the easier it is to:

$$V = \frac{15}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi d\varphi = 2\cdot\frac{15}{4}\int_{0}^{\frac{\pi}{2}} \cos^4\varphi d\varphi = \frac{15}{2}\int_{0}^{\frac{\pi}{2}} \cos^4\varphi d\varphi$$

Now, use a formula from trigonometry:

$$\begin{aligned} &\cos^4\varphi = \cos^2\varphi\cdot\cos^2\varphi = \cos^2\varphi\cdot(1-\sin^2\varphi) = \cos^2\varphi - \sin^2\varphi\cos^2\varphi = \\ &= \cos^2\varphi - \frac{4}{4}\sin^2\varphi\cos^2\varphi = \cos^2\varphi - \frac{1}{4}\sin^2 2\varphi = \frac{1+\cos 2\varphi}{2} - \frac{1}{4}\frac{1-\cos 4\varphi}{2} = \\ &= \frac{1+\cos 2\varphi}{2} - \frac{1-\cos 4\varphi}{8} \end{aligned}$$

Now it is easy to solve this integral ...

We get the solution:

$$\boxed{V = \frac{45\pi}{32}}$$

Example 5.

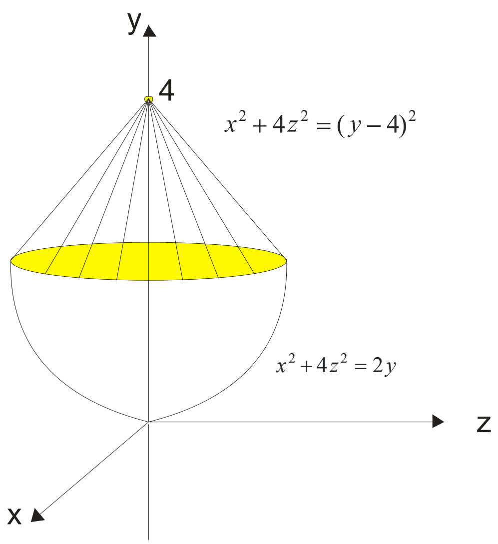
Calculate the volume limited with:

$x^2+4z^2=2y$ and $x^2+4z^2=(y-4)^2$ if $0\le y\le 4$

Solution:

Watch out, bodies are now given along the z - axis but along the y - axis!

It does not change things, thinking the same thing, only slightly correct the formula….



To find cross-sections:

$$
\begin{aligned}
&2y=(y-4)^2\\
&y^2-8y+16-2y=0\\
&y^2-10y+16=0\\
&y_{1,2}=\frac{10\pm\sqrt{100-64}}{2}=\frac{10\pm 6}{2}\to y_1=8\wedge y_2=2
\end{aligned}
$$

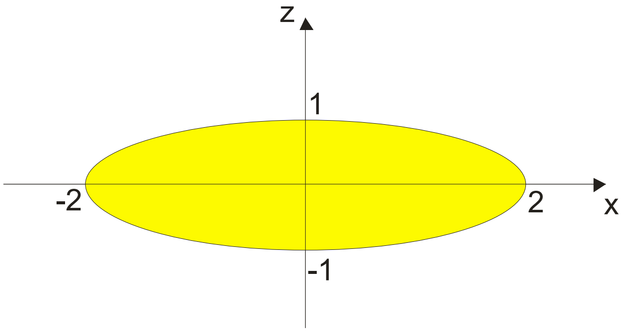
Because $0\le y\le 4$ we take that y = 2.

Then we have :

$$
\begin{aligned}
&x^2+4z^2=2y\wedge y=2\\
&x^2+4z^2=4
\end{aligned}
$$

$\boxed{\frac{x^2}{4}+z^2=1}$ This is our area D in the plane y = 0

We have an ellipse:



We take:

$$\left.\begin{array}{l} x = 2r\cos\varphi \\ z = r\sin\varphi \end{array}\right\} \rightarrow |J| = 2r \qquad \text{then:}$$

$$\frac{x^2}{4} + z^2 = 1 \rightarrow \frac{(2r\cos\varphi)^2}{4} + (r\sin\varphi)^2 = 1 \rightarrow r^2 = 1 \rightarrow \boxed{0 \le r \le 1}$$

$$0 \le \varphi \le 2\pi$$

Express y:

$$\begin{array}{l} x^2 + 4z^2 = (y-4)^2 \\ y - 4 = \pm\sqrt{x^2+4z^2} \\ y_1 = 4 - \sqrt{x^2+4z^2} \end{array}$$

$$\begin{array}{l} x^2 + 4z^2 = 2y \\ y_2 = \dfrac{x^2+4z^2}{2} \end{array}$$

$$V = \iint_D \left(y_1(x,z) - y_2(x,z)\right) dxdz = \iint_D \left(4 - \sqrt{x^2+4z^2} - \frac{x^2+4z^2}{2}\right) dxdz =$$

$$= \iint_D \left(4 - \sqrt{4\left(\frac{x^2}{4}+z^2\right)} - \frac{4\left(\frac{x^2}{4}+z^2\right)}{2}\right) dxdz = \iint_D \left(4 - 2\sqrt{\left(\frac{x^2}{4}+z^2\right)} - 2\left(\frac{x^2}{4}+z^2\right)\right) dxdz$$

polar coordinates:

$$V = \iint_D \left(4 - 2\sqrt{\left(\frac{x^2}{4}+z^2\right)} - 2\left(\frac{x^2}{4}+z^2\right)\right) dxdz = \int_0^{2\pi} d\varphi \int_0^1 \left(4 - r - 2r^2\right) r\,dr =$$

$$= 2\varphi \int_0^1 \left(4r - r^2 - 2r^3\right) dr = 2\varphi \cdot \left(4\frac{r^2}{2} - \frac{r^3}{3} - 2\frac{r^4}{4}\right)\Bigg|_0^1 = 2\varphi \cdot \left(4\frac{1}{2} - \frac{1}{3} - 2\frac{1}{4}\right) = 2\varphi \cdot \left(2 - \frac{1}{3} - \frac{1}{2}\right)$$

$$V = 2\varphi \cdot \frac{5}{6} \rightarrow \boxed{V = \frac{10\pi}{3}}$$