1. Examine function and draw a graph : $y = xe^x$

Domain

This function is everywhere defined , because e^x *is* defined for *all* x in the set R.

So: $x \in (-\infty, \infty)$. This immediately tells us that the function **has no vertical asymptote!**

Zero function

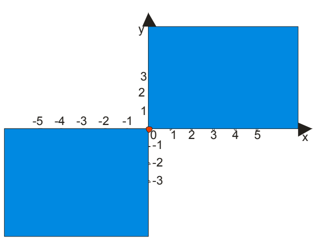
$$y = 0$$
$$xe^x = 0 \rightarrow x = 0$$

To remind you that $e^x > 0$ always.

Sign function

$$y > 0 \rightarrow xe^x > 0 \rightarrow x > 0$$
$$y < 0 \rightarrow xe^x < 0 \rightarrow x < 0$$

The diagram would look:



The function is only in blue field and the x-axis cuts only in x = 0.

Parity

$$f(-x) = -xe^{-x} = \frac{-x}{e^x} \neq f(x)$$

This tells us that the function is neither even nor odd.

Extreme values (max and min) and monotonic function (increasing and decreasing)

$y = xe^x$

$y` = x`e^x + (e^x)`x$

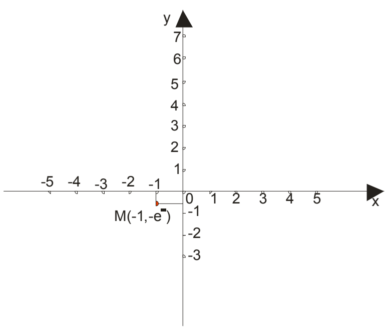
$y` = 1e^x + e^x x$

$y` = e^x(1+x)$

$y` = 0 \rightarrow e^x(1+x) \rightarrow 1+x = 0 \rightarrow x = -1$

For x = -1 is $y = (-1)e^{-1} \rightarrow y = -\frac{1}{e}$

Thus, the extreme point is $M(-1, -\frac{1}{e})$



What the sign of the first derivate depends on ?
As is $e^x > 0$ always , the sign of the first derivate depends on 1+ x

	-∞	-1 ∞
1+x	−	+
y`	− ↘	+ ↗

Point M is then the minimum point.

convexity and concavity

$y` = e^x(1+x)$

$y`` = (e^x)`(1+x) + (1+x)`e^x$

$y`` = e^x(1+x) + e^x$

$y`` = e^x(x+2)$

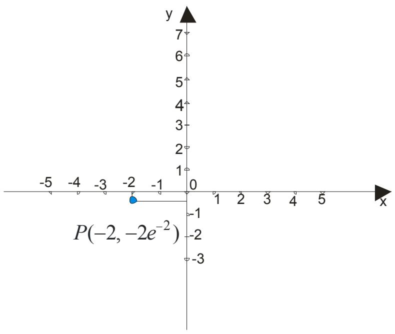
$y`` = 0$

$x + 2 = 0 \rightarrow x = -2$

For x = -2 is $y = -2e^{-2} = \frac{-2}{e^2}$

We have point $P(-2, -2e^{-2})$.

Find that approximately $-2e^{-2} \approx -0,27$



What the sign of the second derivate depends on ?

As $e^x > 0$ then depends on x + 2

	-∞	-2	∞
x+2	–	+	
y``	–	+	

Asymptote function (behavior functions at the ends of the field definition)

As we have said, there is no Vertical asymptote.

Horizontal asymptote

One small tip: If you have a function e^x, especially the work of $x \to +\infty$ and then $x \to -\infty$, because:

$e^{\infty} = \infty$

$e^{-\infty} = 0$

So:

$$\lim_{x\to+\infty} xe^x = \infty \cdot e^{\infty} = \infty \cdot \infty = \infty$$

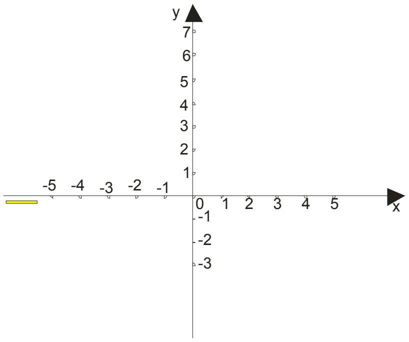
$$\lim_{x\to-\infty} xe^x = -\infty \cdot e^{-\infty} = -\infty \cdot 0 = ?$$

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \frac{-\infty}{e^{-(-\infty)}} = \frac{-\infty}{\infty} = l'Hôpital = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \frac{1}{-\infty} = 0_-$$

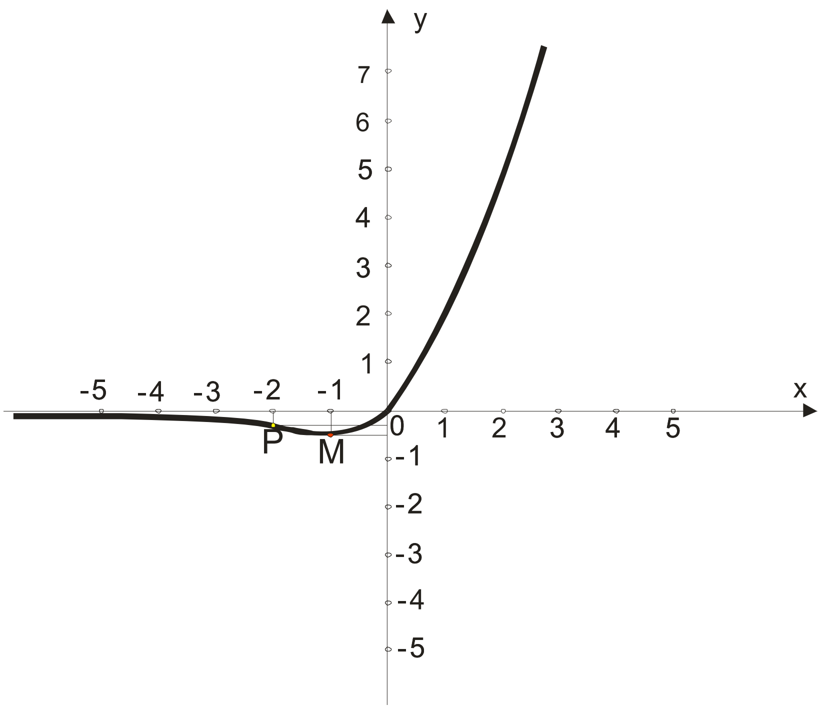
What does this tell us?

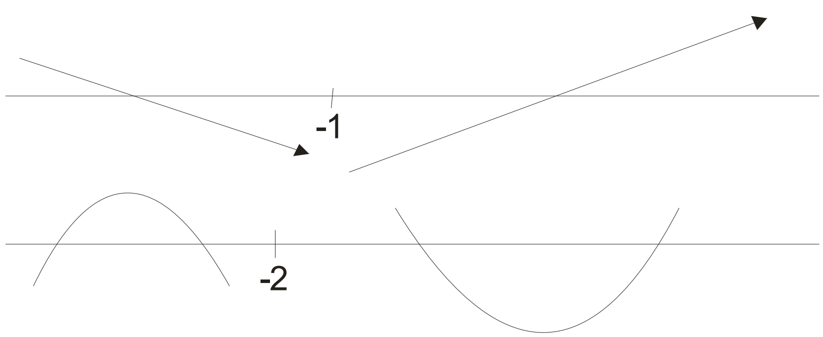
When $x \to +\infty$, there is no horizontal asymptote, but when $x \to -\infty$ horizontal asymptote is y=0, ie,

When x approaches - ∞, the function approaches zero from below, from the negative side!



And to make the final graph:





2. Examine function and draw a graph : $y = \frac{e^x}{x}$

Domain

$x \neq 0 \rightarrow x \in (-\infty, 0) \cup (0, \infty)$

This means that the function in x = 0 has the potential vertical asymptote

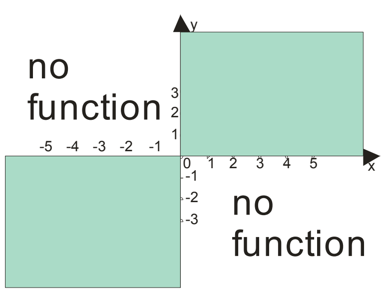
Zero function

As we have said e^x >0 , so the function has no zero (nowhere cuts x axis)

Sign function

It is clear that the sign functions depend only on x

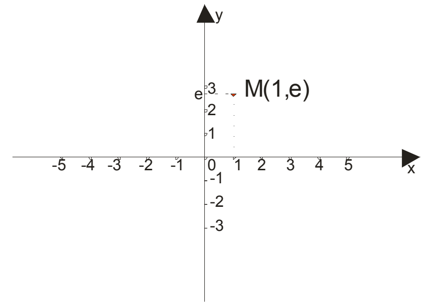
$y > 0 \rightarrow x > 0$
$y < 0 \rightarrow x < 0$



Parity

$$f(-x) = \frac{e^{-x}}{-x} = -\frac{1}{xe^x} \neq f(x)$$

Extreme values (max and min) and monotonic function (increasing and decreasing)



$y=\frac{e^x}{x}$

$y`=\frac{(e^x)`x-x`e^x}{x^2}$

$y`=\frac{e^x x-1e^x}{x^2}$

$y`=\frac{e^x(x-1)}{x^2}$

$y`=0 \rightarrow e^x(x-1)=0 \rightarrow x-1=0 \rightarrow x=1$

<u>For x=1</u> is $y=\frac{e^1}{1} \rightarrow y=e$

$M(1,e)$ is extreme point

As is $x^2>0$ and $e^x>0$ we conclude that the sign of the first derivate depends only on x-1

	-∞ 1	∞
x-1	—	+
y`	—	+
	↘	↗

Point M is then the minimum point!

convexity and concavity

$y`=\frac{e^x(x-1)}{x^2}$

$y``=\frac{[e^x(x-1)]`\cdot x^2-(x^2)`\cdot e^x(x-1)}{x^4}$

$y``=\frac{[(e^x)`(x-1)+(x-1)`e^x]\cdot x^2-2x\cdot e^x(x-1)}{x^4}$

$y``=\frac{[e^x(x-1)+1e^x]\cdot x^2-2x\cdot e^x(x-1)}{x^4}$

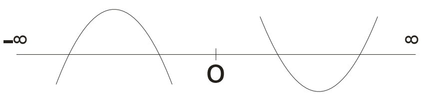
$y``=\frac{[e^x x-1e^x+1e^x]\cdot x^2-2x\cdot e^x(x-1)}{x^4}\ \frac{e^x x\cdot x^2-2x\cdot e^x(x-1)}{x^4}=\frac{e^x \cancel{x}\cdot(x^2-2(x-1))}{x^{\cancel{4}}}$

$y``=\frac{e^x(x^2-2x+2)}{x^3}$

$y`` = 0 \rightarrow x^2 - 2x + 2 = 0$

This quadratic equation has no solution, because it is D<0 and $a > 0$.

So: $x^2 - 2x + 2 > 0$

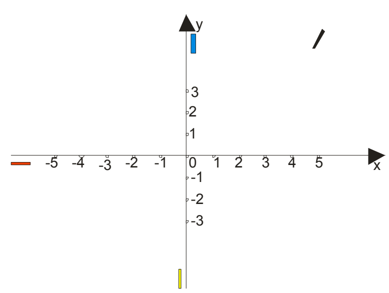


Asymptote function (behavior functions at the ends of the field definition)

Vertical asymptote

$$\lim_{x\to 0+\varepsilon} \frac{e^x}{x} = \frac{e^0}{0+\varepsilon} = \frac{1}{+\varepsilon} = +\infty \qquad \text{(Blue line)}$$

$$\lim_{x\to 0-\varepsilon} \frac{e^x}{x} = \frac{e^0}{0-\varepsilon} = \frac{1}{-\varepsilon} = -\infty \qquad \text{(Yellow line)}$$



Horizontal asymptote

$$\lim_{x\to+\infty} \frac{e^x}{x} = \frac{e^\infty}{\infty} = \frac{\infty}{\infty} = l'Hôpital = \lim_{x\to+\infty} \frac{(e^x)`}{x`} = \lim_{x\to+\infty} \frac{e^x}{1} = e^\infty = \infty \qquad \text{(Black line)}$$

$$\lim_{x\to-\infty} \frac{e^x}{x} = \frac{e^{-\infty}}{-\infty} = \frac{0}{-\infty} = 0\_ \qquad \text{(Red line)}$$

Therefore, the function has a horizontal asymptote y = 0 but only on the left

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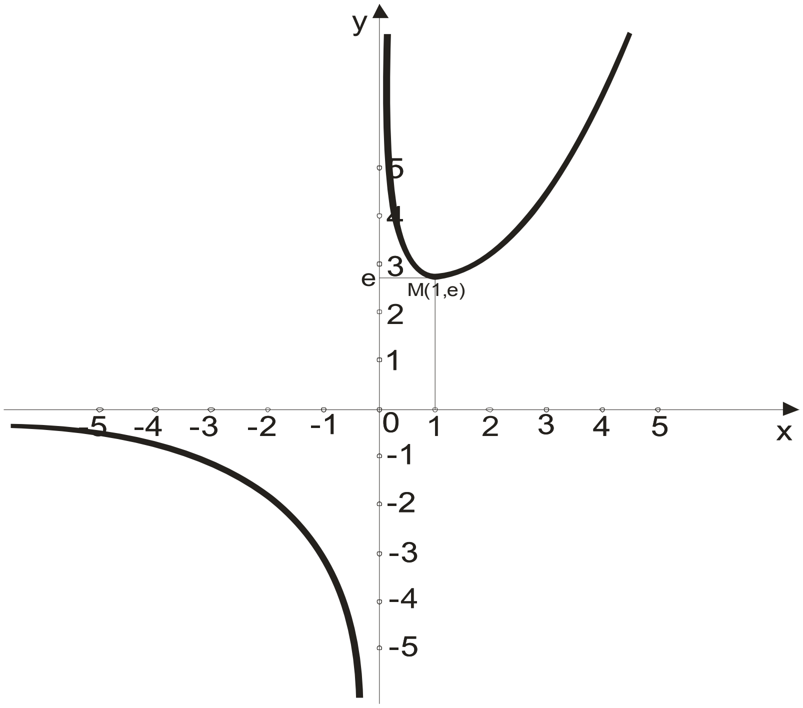
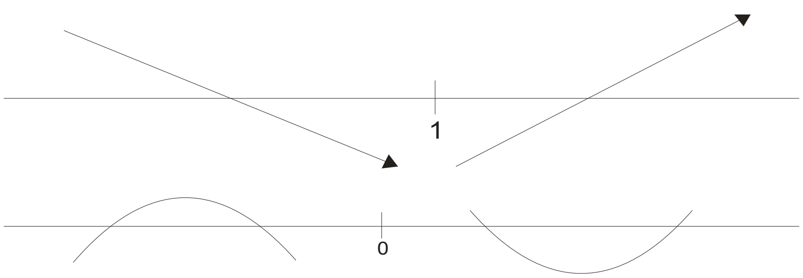
The final graph looks like:

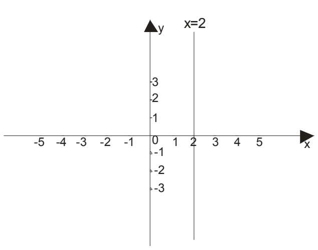




3. Examine function and draw a graph : $y = x \cdot e^{\frac{1}{x-2}}$

Domain

$x-2 \neq 0 \rightarrow x \neq 2 \rightarrow x \in (-\infty, 2) \cup (2, \infty)$



Zero function

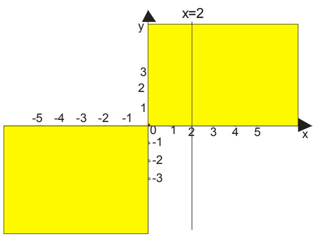
$$y = 0 \rightarrow x \cdot e^{\frac{1}{x-2}} = 0 \rightarrow x = 0 \quad \text{because} \quad e^{\frac{1}{x-2}} > 0 \ \text{always}$$

Sign function

As is $e^{\frac{1}{x-2}} > 0$ **,** concludes that the sign function depends only on x-2

y > 0 when x – 2 > 0 , then x > 2

y < 0 when x – 2 < 0 , then x < 2



The function is only found in the yellow areas.

Parity

$$f(-x) = -x \cdot e^{\frac{1}{-x-2}} \neq f(x)$$

Extreme values (max and min) and monotonic function (increasing and decreasing)

$$y = x \cdot e^{\frac{1}{x-2}} \quad \text{...} (e^{\Theta})` = e^{\Theta} \cdot \Theta`$$

$$y` = 1 \cdot e^{\frac{1}{x-2}} + (e^{\frac{1}{x-2}})` \cdot x$$

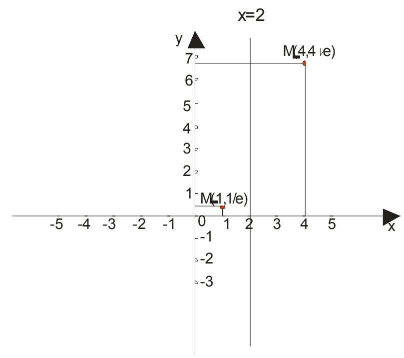
$$y` = e^{\frac{1}{x-2}} + e^{\frac{1}{x-2}} \cdot (\frac{1}{x-2})` \cdot x$$

$$y` = e^{\frac{1}{x-2}} + e^{\frac{1}{x-2}} \cdot (-\frac{1}{(x-2)^2}) \cdot x = e^{\frac{1}{x-2}} - e^{\frac{1}{x-2}} \cdot \frac{1}{(x-2)^2} \cdot x = e^{\frac{1}{x-2}} \cdot (1 - \frac{x}{(x-2)^2}) = e^{\frac{1}{x-2}} \cdot \frac{(x-2)^2 - x}{(x-2)^2}$$

$$y` = e^{\frac{1}{x-2}} \cdot \frac{x^2 - 4x + 4 - x}{(x-2)^2}$$

$$y` = e^{\frac{1}{x-2}} \cdot \frac{x^2 - 5x + 4}{(x-2)^2}$$

$$y` = 0 \rightarrow x^2 - 5x + 4 = 0 \rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow x_1 = 1; x_2 = 4$$



For x = 1

$$y = 1 \cdot e^{\frac{1}{1-2}} = e^{-1} = \frac{1}{e} \rightarrow M_1 = (1, \frac{1}{e})$$

For x = 4

$$y = 4 \cdot e^{\frac{1}{4-2}} = 4e^{\frac{1}{2}} = 4\sqrt{e} \rightarrow M_2 = (4, 4\sqrt{e})$$

What the sign of the first derivate depends on ?

As is $e^{\frac{1}{x-2}} > 0$ **and** $(x-2)^2 > 0$**,** sign of the first derivate depends on $x^2 - 5x + 4$

	-∞	1	4	∞
x-1	−	+	+	
x-4	−	−	+	
y`	+ ↗	− ↘	+ ↗	

convexity and concavity

$$y` = e^{\frac{1}{x-2}} \cdot \frac{x^2 - 5x + 4}{(x-2)^2}$$

$$y`` = (e^{\frac{1}{x-2}})` \cdot \frac{x^2 - 5x + 4}{(x-2)^2} + (\frac{x^2 - 5x + 4}{(x-2)^2})` e^{\frac{1}{x-2}}$$

$$y`` = e^{\frac{1}{x-2}} \cdot (-\frac{1}{(x-2)^2}) \cdot \frac{x^2 - 5x + 4}{(x-2)^2} + \frac{(x^2 - 5x + 4)` (x-2)^2 - ((x-2)^2)` (x^2 - 5x + 4)}{(x-2)^4} \cdot e^{\frac{1}{x-2}}$$

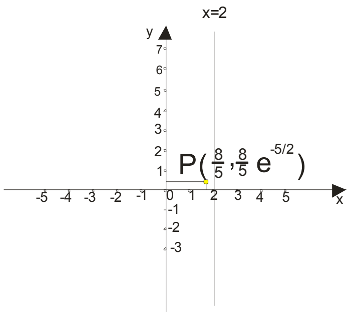
We get :

$$y`` = \frac{5x - 8}{(x-2)^4} \cdot e^{\frac{1}{x-2}}$$

$$y``=0 \to 5x-8=0 \to x=\frac{8}{5}$$

For $x=\frac{8}{5} \to y=\frac{8}{5}\cdot e^{\frac{1}{\frac{8}{5}-2}} \to y=\frac{8}{5}\cdot e^{\frac{1}{-\frac{2}{5}}} \to y=\frac{8}{5}\cdot e^{-\frac{5}{2}}$

So : $P(\frac{8}{5},\frac{8}{5}\cdot e^{-\frac{5}{2}})$



What the sign of the second derivate depends on ?

Sign of the second derivate depends on 5x-8

	-∞	$\frac{8}{5}$ ∞
5x-8	−	+
y``	−	+

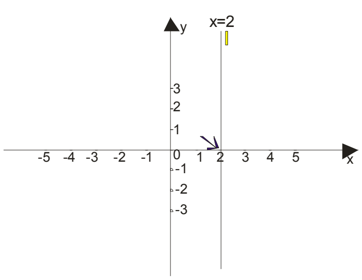
Asymptote function (behavior functions at the ends of the field definition)

Vertical asymptote

$$y=x\cdot e^{\frac{1}{x-2}}$$

$$\lim_{x\to 2+\varepsilon} xe^{\frac{1}{x-2}}=2\cdot e^{\frac{1}{2+\varepsilon-2}}=2\cdot e^{\frac{1}{+\varepsilon}}=2\cdot e^{\infty}=\infty$$ (Yellow line)

$$\lim_{x\to 2-\varepsilon} xe^{\frac{1}{x-2}}=2\cdot e^{\frac{1}{2-\varepsilon-2}}=2\cdot e^{\frac{1}{-\varepsilon}}=2\cdot e^{-\infty}=2\cdot 0=0$$ (blue arrow)



Horizontal asymptote

$$\lim_{x\to+\infty} xe^{\frac{1}{x-2}} = \infty\cdot e^{\frac{1}{\infty-2}} = \infty\cdot e^{\frac{1}{\infty}} = \infty\cdot e^{0} = \infty\cdot 1 = \infty$$

$$\lim_{x\to-\infty} xe^{\frac{1}{x-2}} = -\infty\cdot e^{\frac{1}{-\infty-2}} = -\infty\cdot e^{\frac{1}{-\infty}} = -\infty\cdot e^{0} = -\infty\cdot 1 = -\infty$$

No horizontal asymptotes, and we must examine whether there is oblique asymptote **...**

Oblique asymptote

y= kx+n

$$k = \lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to\pm\infty}\frac{\cancel{x}e^{\frac{1}{x-2}}}{\cancel{x}} = \lim_{x\to\pm\infty} e^{\frac{1}{x-2}} = e^{\frac{1}{\infty-2}} = e^{0} = 1$$

$$n = \lim_{x\to\pm\infty}[f(x)-kx] = \lim_{x\to\pm\infty}[xe^{\frac{1}{x-2}} - 1\cdot x] = \lim_{x\to\pm\infty} x(e^{\frac{1}{x-2}} - 1) = \infty\cdot 0 = ?$$

$$= \lim_{x\to\pm\infty}\frac{e^{\frac{1}{x-2}}-1}{\frac{1}{x}} = \frac{0}{0} = \lim_{x\to\pm\infty}\frac{e^{\frac{1}{x-2}}\cdot(-\frac{1}{(x-2)^2})}{-\frac{1}{x^2}} = \lim_{x\to\pm\infty} e^{\frac{1}{x-2}}\cdot\frac{x^2}{(x-2)^2} = \lim_{x\to\pm\infty} e^{\frac{1}{x-2}}\cdot\lim_{x\to\pm\infty}\frac{x^2}{(x-2)^2} = 1\cdot 1 = 1$$

We have oblique asymptote **:**

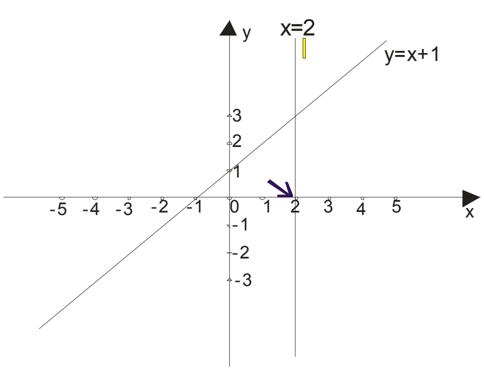
y= kx + n so: y = x + 1

For x=0

$y = 0+1 = 1$

For y=0

$0 = x+1 \rightarrow x = -1$



x	0	-1
y	1	0

And to conclude the final graph: