Extremes of functions with multiple variables (part I)

In most of these tasks is set of functions $f(x,y)$.

The first job is to find partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Then solve the system of equations $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

Solutions of this system (may be one, but several of them) give us a **stationary points** (x_0,y_0), (x_1,y_1), ...etc.

Next we find $A=\frac{\partial^2 f}{\partial x^2}$, $B=\frac{\partial^2 f}{\partial x \partial y}$, $C=\frac{\partial^2 f}{\partial y^2}$

Organize function $D = A \cdot C - B^2$

For each stationary point we doing the same thing:

First, replace the value of **stationary points** in *D*. The resulting value must be greater than zero $\boxed{D>0}$,

If it occurs it is not, then that point is not the point of extreme.

Further investigate whether the maximum or minimum:

i) If $\boxed{D>0}$ and $A<0$ $(C<0)$ then the stationary point is **maximum**

ii) If $\boxed{D>0}$ and $A>0$ $(C>0)$ then the stationary point is **minimum**

iii) If $D=0$ case is **indeterminate**

iv) If $D<0$ then **no extreme**

If it happens that $D=0$, that is, that the case is vague, we have to go to a broader definition of extreme , that is,

looking for second-order differentia $d^2 f$ and

a) If $d^2 f>0$ function has a minimum

b) If $d^2 f<0$ function has a maximum

Example 1.

Find extreme for function $z = x^3 + 8y^3 - 6xy + 5$

Solution:

First, look for the first partial derivatives:

$$z = x^3 + 8y^3 - 6xy + 5$$
$$\frac{\partial z}{\partial x} = 3x^2 - 6y$$
$$\frac{\partial z}{\partial y} = 24y^2 - 6x$$

Establish a system of equations $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$.

$$3x^2 - 6y = 0 \text{.....} / : 3$$
$$\underline{24y^2 - 6x = 0} \text{......} / : 6$$
$$x^2 - 2y = 0 \rightarrow y = \frac{x^2}{2} \quad \text{replace in} \quad 4y^2 - x = 0$$
$$\underline{4y^2 - x = 0}$$
$$4\left(\frac{x^2}{2}\right)^2 - x = 0 \rightarrow \cancel{4}\frac{x^4}{\cancel{4}} - x = 0 \rightarrow x^4 - x = 0 \rightarrow x(x^3 - 1) = 0$$
$$x = 0 \vee x = 1$$

For $x = 0 \rightarrow y = \frac{x^2}{2} \rightarrow y = 0 \rightarrow \boxed{M_1(0,0)}$

For $x = 1 \rightarrow y = \frac{x^2}{2} \rightarrow y = \frac{1}{2} \rightarrow \boxed{M_1(1,\frac{1}{2})}$

We got two stationary points: $M_1(0,0)$ and $M_1(1,\frac{1}{2})$.

Next, we must find $A = \frac{\partial^2 z}{\partial x^2}$, $B = \frac{\partial^2 z}{\partial x \partial y}$, $C = \frac{\partial^2 z}{\partial y^2}$ and make $D = A \cdot C - B^2$

$$\frac{\partial z}{\partial x} = 3x^2 - 6y$$

$$\frac{\partial z}{\partial y} = 24y^2 - 6x$$

$$A = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 6y) = 6x$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(24y^2 - 6x) = -6$$

$$C = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(24y^2 - 6x) = 48y$$

then is :

$$D = A \cdot C - B^2$$

$$D = (6x)(48y) - (-6)^2$$

$$\boxed{D = 288xy - 36}$$

Taking the first stationary point and investigate:

$M_1(0,0)$

$$D = 288xy - 36 \to D(0,0) = 288 \cdot 0 \cdot 0 - 36 = -36 \to D(0,0) < 0$$

Therefore, we have shown that this point is not extreme!

Investigate the second stationary point:

$M_1(1, \frac{1}{2})$

$$D = 288xy - 36 \to D(1, \frac{1}{2}) = 288 \cdot 1 \cdot \frac{1}{2} - 36 = 144 - 36 = 108 \to D(1, \frac{1}{2}) > 0$$

This point is extreme, even to investigate whether the max or min.

$$A = 6x \to A(1, \frac{1}{2}) = 6 \cdot 1 = 6 \to A(1, \frac{1}{2}) > 0$$

We conclude that this point is minimum!

This value back to the starting function to calculate the minimum value:

$$z = x^3 + 8y^3 - 6xy + 5$$

$$z_{\min}(1, \frac{1}{2}) = 1^3 + 8(\frac{1}{2})^3 - 6 \cdot 1 \cdot (\frac{1}{2}) + 5 = 1 + 1 - 3 + 5 = 4$$

$$\boxed{z_{\min}(1, \frac{1}{2}) = 4}$$

Example 2.

Find extreme for function : $z = x\sqrt{y} - x^2 - y + 6x + 3$

Solution:

$$z = x\sqrt{y} - x^2 - y + 6x + 3$$
$$\frac{\partial z}{\partial x} = \sqrt{y} - 2x + 6$$
$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}} - 1$$

$$\sqrt{y} - 2x + 6 = 0$$
$$\frac{x}{2\sqrt{y}} - 1 = 0$$

Iz $\frac{x}{2\sqrt{y}} - 1 = 0 \rightarrow x = 2\sqrt{y}$, then this replace in $\sqrt{y} - 2x + 6 = 0$

$$\sqrt{y} - 2 \cdot 2\sqrt{y} + 6 = 0$$
$$-3\sqrt{y} + 6 = 0 \rightarrow \sqrt{y} = 2 \rightarrow \boxed{y = 4}$$
$$x = 2\sqrt{y} \rightarrow x = 2 \cdot \sqrt{4} \rightarrow \boxed{x = 4}$$

$\boxed{M(4,4)}$ is stationary point (only)

Still looking $A = \frac{\partial^2 z}{\partial x^2}$, $B = \frac{\partial^2 z}{\partial x \partial y}$, $C = \frac{\partial^2 z}{\partial y^2}$ and $D = A \cdot C - B^2$

$$\frac{\partial z}{\partial x} = \sqrt{y} - 2x + 6$$
$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}} - 1$$
$$A = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\sqrt{y} - 2x + 6) = -2$$
$$B = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{x}{2\sqrt{y}} - 1) = \frac{1}{2\sqrt{y}}$$
$$C = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{x}{2\sqrt{y}} - 1) = \frac{\partial}{\partial y}(\frac{x}{2} \cdot y^{-\frac{1}{2}} - 1) = -\frac{x}{4y\sqrt{y}}$$

$$D = A \cdot C - B^2$$

Then is : $$D = (-2)(-\frac{x}{4y\sqrt{y}}) - (\frac{1}{2\sqrt{y}})^2$$

$$\boxed{D = \frac{x}{2y\sqrt{y}} - \frac{1}{4y}}$$

The value of point $M(4,4)$ replace in D :

$$D = \frac{x}{2y\sqrt{y}} - \frac{1}{4y}$$

$$D(4,4) = \frac{4}{2 \cdot 4\sqrt{4}} - \frac{1}{4 \cdot 4} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

$$D(4,4) = \frac{3}{16} > 0$$

We conclude that our point $M(4,4)$ is extreme.

As is $A = -2$, we conclude A = -2 < 0 , and point $M(4,4)$ is maximum!

Return to $z = x\sqrt{y} - x^2 - y + 6x + 3$ to find the maximum value:

$$z = x\sqrt{y} - x^2 - y + 6x + 3$$

$$z(4,4) = 4\sqrt{4} - 4^2 - 4 + 6 \cdot 4 + 3$$

$$\boxed{z(4,4) = 15}$$

Example 3.

Find the extreme values of function : $z = 3\ln\frac{x}{6} + 2\ln y + \ln(12 - x - y)$

Solution:

$$z = 3\ln\frac{x}{6} + 2\ln y + \ln(12 - x - y)$$

$$\frac{\partial z}{\partial x} = 3 \cdot \frac{1}{\frac{x}{6}} \cdot \frac{1}{6} + \frac{1}{12 - x - y}(-1) \rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{3}{x} - \frac{1}{12 - x - y}}$$

$$\frac{\partial z}{\partial y} = 2 \cdot \frac{1}{y} + \frac{1}{12 - x - y}(-1) \rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{2}{y} - \frac{1}{12 - x - y}}$$

Equate the first partial derivatives to zero and solve the system of equations:

$$\frac{3}{x}-\frac{1}{12-x-y}=0$$
$$\frac{2}{y}-\frac{1}{12-x-y}=0$$

$$\frac{3}{x}=\frac{1}{12-x-y}\rightarrow 4x+3y=36$$
$$\frac{2}{y}=\frac{1}{12-x-y}\rightarrow 2x+3y=24$$

$$4x+3y=36$$
$$2x+3y=24$$

$$x=6$$
$$y=4$$
$$\boxed{M(6,4)}$$

We got a stationary point.

We are looking for $A=\frac{\partial^2 z}{\partial x^2}$, $B=\frac{\partial^2 z}{\partial x \partial y}$, $C=\frac{\partial^2 z}{\partial y^2}$ and $D=A\cdot C-B^2$

$$\frac{\partial z}{\partial x}=\frac{3}{x}-\frac{1}{12-x-y}$$
$$\frac{\partial z}{\partial y}=\frac{2}{y}-\frac{1}{12-x-y}$$

$$A=\frac{\partial^2 z}{\partial x^2}=-\frac{3}{x^2}-\frac{1}{(12-x-y)^2}$$
$$B=\frac{\partial^2 z}{\partial x \partial y}=-\frac{1}{(12-x-y)^2}$$
$$C=\frac{\partial^2 z}{\partial y^2}=-\frac{2}{y^2}-\frac{1}{(12-x-y)^2}$$

$$D=A\cdot C-B^2$$
$$D=(-\frac{3}{x^2}-\frac{1}{(12-x-y)^2})\cdot(-\frac{2}{y^2}-\frac{1}{(12-x-y)^2})-\frac{1}{(12-x-y)^4}$$

Substituting the value of fixed points in D:

$$M(6,4) \rightarrow D = (-\frac{3}{x^2} - \frac{1}{(12-x-y)^2}) \cdot (-\frac{2}{y^2} - \frac{1}{(12-x-y)^2}) - \frac{1}{(12-x-y)^4}$$

$$D(6,4) = (-\frac{3}{6^2} - \frac{1}{(12-6-4)^2}) \cdot (-\frac{2}{4^2} - \frac{1}{(12-6-4)^2}) - \frac{1}{(12-6-4)^4}$$

$$D(6,4) = \frac{1}{8} > 0$$

Now change this point in A to determine whether the max or min:

$$M(6,4) \rightarrow A = -\frac{3}{x^2} - \frac{1}{(12-x-y)^2}$$

$$A(6,4) = -\frac{3}{6^2} - \frac{1}{(12-6-4)^2} = -\frac{1}{12} - \frac{1}{4} < 0$$

Our point M(6,4) is therefore maximum!

The value function in it is:

$$z = 3\ln\frac{x}{6} + 2\ln y + \ln(12-x-y)$$

$$z(6,4) = 3\ln\frac{6}{6} + 2\ln 4 + \ln(12-6-4)$$

$$z(6,4) = 3\ln 1 + 2\ln 2^2 + \ln 2 = 3 \cdot 0 + 2 \cdot 2\ln 2 + \ln 2$$

$$\boxed{z(6,4) = 5\ln 2}$$