SOLVING THE SYSTEM EQUATIONS (METHOD Determinants)

In this file we will try to give you explain how to apply **determinants** to solving systems of linear equations.

It is important to note here that we observe only the quadratic systems $S_{n\times n}$. That is, systems that have the same number of unknowns and equations.

Professors often associated systems $S_{3\times 3}$ or $S_{4\times 4}$. And we will devote attention to them.

The system can be **homogeneous** and **inhomogeneous.**

First, a look inhomogeneous system $S_{3\times 3}$ (Three equations, three unknown)

$$\begin{aligned} a_1x+b_1y+c_1z&=t_1\\ a_2x+b_2y+c_2z&=t_2\\ a_3x+b_3y+c_3z&=t_3 \end{aligned}$$

From here, first forming **the determinant of** getting the numbers in front of the unknown: $D=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3 \end{vmatrix}$

Then replace members infront ***x* with** free members (from right of equality): $D_x=\begin{vmatrix} t_1 & b_1 & c_1\\ t_2 & b_2 & c_2\\ t_3 & b_3 & c_3 \end{vmatrix}$

Replace members infront ***y*** **with** free members: $D_y=\begin{vmatrix} a_1 & t_1 & c_1\\ a_2 & t_2 & c_2\\ a_3 & t_3 & c_3 \end{vmatrix}$

Replace members infront ***z*** **with** free members: : $D_z=\begin{vmatrix} a_1 & b_1 & t_1\\ a_2 & b_2 & t_2\\ a_3 & b_3 & t_3 \end{vmatrix}$

In this way we get four determinants:

$$D=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3 \end{vmatrix}\quad D_x=\begin{vmatrix} t_1 & b_1 & c_1\\ t_2 & b_2 & c_2\\ t_3 & b_3 & c_3 \end{vmatrix}\quad D_y=\begin{vmatrix} a_1 & t_1 & c_1\\ a_2 & t_2 & c_2\\ a_3 & t_3 & c_3 \end{vmatrix}\quad D_z=\begin{vmatrix} a_1 & b_1 & t_1\\ a_2 & b_2 & t_2\\ a_3 & b_3 & t_3 \end{vmatrix}$$

In every task our first job is to find values for these determinations.

Further solutions search using **Cramer's theorem:**

i) **If the determinant of system** $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ **different from zero, then the system has a unique solution,**

search by: $x = \frac{D_x}{D};\quad y = \frac{D_y}{D};\quad z = \frac{D_z}{D}$

ii) **If the determinant of system** $D = 0$ **and** $D_x = D_y = D_z = 0$ **system has infinitely many solutions**

(Is undetermined)

iii) **If the determinant of system** $D = 0$ **and** $D_x \neq 0 \vee D_y \neq 0 \vee D_z \neq 0$ **(meaning, at least one of these three determinants that is different from zero) the system is impossible. (No solution)**

Take care, all this applies to the inhomogeneous system!

What if we have a homogeneous system?

If we look homogenous system :
$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

It is clear that he still has the **trivial** solution : $(x, y, z) = (0, 0, 0)$

Square homogeneous system has a trivial solution if and only if $\underline{D = 0}$

So, to our homogeneous system must be: $\boxed{D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0}$

Thinking for s ystems $S_{4\times 4}$ (4 equations, 4 unknowns) is completely analogous to this.

Consider the system: $\begin{array}{l} a_1x+b_1y+c_1z+d_1t=u_1 \\ a_2x+b_2y+c_2z+d_2t=u_2 \\ a_3x+b_3y+c_3z+d_3t=u_3 \\ a_4x+b_4y+c_4z+d_4t=u_4 \end{array}$. Here we seek for the following determinations:

$$D=\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \quad D_x=\begin{vmatrix} u_1 & b_1 & c_1 & d_1 \\ u_2 & b_2 & c_2 & d_2 \\ u_3 & b_3 & c_3 & d_3 \\ u_4 & b_4 & c_4 & d_4 \end{vmatrix} \quad D_y=\begin{vmatrix} a_1 & u_1 & c_1 & d_1 \\ a_2 & u_2 & c_2 & d_2 \\ a_3 & u_3 & c_3 & d_3 \\ a_4 & u_4 & c_4 & d_4 \end{vmatrix} \quad D_z=\begin{vmatrix} a_1 & b_1 & u_1 & d_1 \\ a_2 & b_2 & u_2 & d_2 \\ a_3 & b_3 & u_3 & d_3 \\ a_4 & b_4 & u_4 & d_4 \end{vmatrix} \quad D_t=\begin{vmatrix} a_1 & b_1 & c_1 & u_1 \\ a_2 & b_2 & c_2 & u_2 \\ a_3 & b_3 & c_3 & u_3 \\ a_4 & b_4 & c_4 & u_4 \end{vmatrix}$$

Then:

$x=\frac{D_x}{D}; \quad y=\frac{D_y}{D}; \quad z=\frac{D_z}{D}; \quad t=\frac{D_t}{D}$, further solutions search using **Cramer's theorem…**

If we have the homogeneous system:

$$\begin{array}{l} a_1x+b_1y+c_1z+d_1t=0 \\ a_2x+b_2y+c_2z+d_2t=0 \\ a_3x+b_3y+c_3z+d_3t=0 \\ a_4x+b_4y+c_4z+d_4t=0 \end{array}$$

then must be $D=0$.

TASKS

1. Solve the system of equations:

$$\begin{array}{l} x+2y-5z=6 \\ 2x+y+2z=5 \\ -3x+3y-4z=8 \end{array}$$

<u>Solution:</u>

Of course, this system is much easier to solve the Gauss method or another, but as we study determinants.

On this occasion, we go through difficult:

Calculates the value of the following determinants (we will use **Sarusovo rule of** correspondence with the first two Columns)

$$D = \begin{vmatrix} 1 & 2 & -5 \\ 2 & 1 & 2 \\ -3 & 3 & -4 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & -5 \\ 2 & 1 & 2 \\ -3 & 3 & -4 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 1 \\ -3 & 3 \end{matrix} = -4 - 12 - 30 + 16 - 6 - 15 = -51 \rightarrow \boxed{D = -51}$$

Since the determinant of the system is different from zero, we know immediately that the system will have a unique solution!

$$D_x = \begin{vmatrix} 6 & 2 & -5 \\ 5 & 1 & 2 \\ 8 & 3 & -4 \end{vmatrix}$$

$$\begin{vmatrix} 6 & 2 & -5 \\ 5 & 1 & 2 \\ 8 & 3 & -4 \end{vmatrix} \begin{matrix} 6 & 2 \\ 5 & 1 \\ 8 & 3 \end{matrix} = -24 + 32 - 75 + 40 - 36 + 40 = -23 \rightarrow \boxed{D_x = -23}$$

$$D_y = \begin{vmatrix} 1 & 6 & -5 \\ 2 & 5 & 2 \\ -3 & 8 & -4 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 6 & -5 \\ 2 & 5 & 2 \\ -3 & 8 & -4 \end{vmatrix} \begin{matrix} 1 & 6 \\ 2 & 5 \\ -3 & 8 \end{matrix} = -20 - 36 - 80 + 48 - 16 - 75 = -179 \rightarrow \boxed{D_y = -179}$$

$$D_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 1 & 5 \\ -3 & 3 & 8 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 6 \\ 2 & 1 & 5 \\ -3 & 3 & 8 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 1 \\ -3 & 3 \end{matrix} = 8 - 30 + 36 - 32 - 15 + 18 = -15 \rightarrow \boxed{D_z = -15}$$

$$x = \frac{D_x}{D} = \frac{-15}{-51} = \frac{5}{17}$$

$$y = \frac{D_y}{D} = \frac{-179}{-51} = \frac{179}{51}$$

$$z = \frac{D_z}{D} = \frac{-23}{-51} = \frac{23}{51}$$

2. Depending on the parameter *a*, to discuss and solve the system:

$$\begin{aligned} ax+y+z&=1\\ x+ay+z&=a\\ x+y+az&=a^2 \end{aligned}$$

Solution:

$$D=\begin{vmatrix} a & 1 & 1\\ 1 & a & 1\\ 1 & 1 & a \end{vmatrix}$$

$$\begin{vmatrix} a & 1 & 1\\ 1 & a & 1\\ 1 & 1 & a \end{vmatrix}\begin{matrix} a & 1\\ 1 & a\\ 1 & 1 \end{matrix}=a^3+1+1-a-a-a=a^3-3a+2$$

$$a^3-3a+2=a^3-a-2a+2=a(a^2-1)-2(a-1)=a(a-1)(a+1)-2(a-1)=(a-1)(a^2+a-2)=$$
$$=(a-1)(a-1)(a+2)=(a-1)^2(a+2)$$

$$\boxed{D=(a-1)^2(a+2)}$$

There is not enough to find value of the determinants, but that solution must be "pack" .

$$D_x=\begin{vmatrix} 1 & 1 & 1\\ a & a & 1\\ a^2 & 1 & a \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1\\ a & a & 1\\ a^2 & 1 & a \end{vmatrix}\begin{matrix} 1 & 1\\ a & a\\ a^2 & 1 \end{matrix}=a^2+a^2+a-a^2-1-a^3=-a^3+a^2+a-1=-a^2(a-1)+1(a-1)=$$

$$(a-1)(-a^2+1)=(a-1)(1-a)(1+a)=-(a-1)(a-1)(1+a)=-(a-1)^2(a+1)$$

$$\boxed{D_x=-(a-1)^2(a+1)}$$

$$D_y=\begin{vmatrix} a & 1 & 1\\ 1 & a & 1\\ 1 & a^2 & a \end{vmatrix}$$

$$\begin{vmatrix} a & 1 & 1\\ 1 & a & 1\\ 1 & a^2 & a \end{vmatrix}\begin{matrix} a & 1\\ 1 & a\\ 1 & a^2 \end{matrix}=a^3+1+a^2-a-a^3-a=a^2-2a+1=(a-1)^2$$

$$\boxed{D_y=(a-1)^2}$$

$$D_z = \begin{vmatrix} a & 1 & 1 \\ 1 & a & a \\ 1 & 1 & a^2 \end{vmatrix}$$

$$\begin{vmatrix} a & 1 & 1 \\ 1 & a & a \\ 1 & 1 & a^2 \end{vmatrix} \begin{matrix} a & 1 \\ 1 & a \\ 1 & 1 \end{matrix} = a^4 + a + 1 - a^2 - a^2 - a = a^4 - 2a^2 + 1 = (a^2-1)^2 = (a-1)^2(a+1)^2$$

$$\boxed{D_z = (a-1)^2(a+1)^2}$$

We've completed the technical part of the job, find solutions and packed them. Our advice is to copy them now, because of the following discussion:

$$D = (a-1)^2(a+2)$$
$$D_x = -(a-1)^2(a+1)$$
$$D_y = (a-1)^2$$
$$D_z = (a-1)^2(a+1)^2$$

Kramer says that the system has a unique solution if $D \neq 0$.

In this case, must be:

$$D \neq 0 \to (a-1)^2(a+2) \neq 0 \to a \neq 1 \wedge a \neq -2$$

If $a \neq 1 \wedge a \neq -2$ system has a unique solution:

$$x = \frac{D_x}{D} = \frac{-\cancel{(a-1)^2}(a+1)}{\cancel{(a-1)^2}(a+2)} = -\frac{a+1}{a+2} \to \boxed{x = -\frac{a+1}{a+2}}$$

$$y = \frac{D_y}{D} = \frac{\cancel{(a-1)^2}}{\cancel{(a-1)^2}(a+2)} = \frac{1}{a+2} \to \boxed{y = \frac{1}{a+2}}$$

$$z = \frac{D_z}{D} = \frac{\cancel{(a-1)^2}(a+1)^2}{\cancel{(a-1)^2}(a+2)} = \frac{(a+1)^2}{a+2} \to \boxed{z = \frac{(a+1)^2}{a+2}}$$

But this job is not finished, because we have to examine what happens if $a = 1$, and then if $a = -2$.

for $a=1$

$$D=(a-1)^2(a+2)=(1-1)^2(1+2)=0$$
$$D_x=-(a-1)^2(a+1)=-(1-1)^2(1+1)=0$$
$$D_y=(a-1)^2=(1-1)^2=0$$
$$D_z=(a-1)^2(a+1)^2=(1-1)^2(1+1)^2=0$$

According to Kramer this system has infinitely many solutions, we return to the starting system and replace $a=1$.

$$\boxed{\begin{matrix} ax+y+z=1 \\ x+ay+z=a \\ x+y+az=a^2 \end{matrix}} \rightarrow \boxed{\begin{matrix} 1x+y+z=1 \\ x+1y+z=1 \\ x+y+1z=1^2 \end{matrix}} \rightarrow \boxed{\begin{matrix} x+y+z=1 \\ x+y+z=1 \\ x+y+z=1 \end{matrix}}$$

$$x+y+z=1 \rightarrow z=1-x-y$$

The system is undetermined and describe solutions with $(x,y,z)=(x,y,1-x-y)$ $x\in R, y\in R$

Note: Some professors require to introduce a new letter (letters) describe the solution, for example:

$(p,q,1-p-q)$ $p\in R, q\in R$

Our advice is as always the same: to work as required by your professor…

for $a=-2$

$$D=(a-1)^2(a+2)=(-2-1)^2(-2+2)=0$$
$$D_x=-(a-1)^2(a+1)=-(-2-1)^2(-2+1)=-9\cdot(-1)=9\rightarrow \boxed{D_x\neq 0}$$

We do not have to change further, according to Kramer, this system is **impossible.**

3. Depending on the parameter *a*, to discuss and solve the system:

$$\begin{aligned} x+y+z&=0 \\ ax+4y+z&=0 \\ 6x+(a+2)y+2z&=0 \end{aligned}$$

Solution:

First, we notice that the system is homogeneous, that is, always has a solution $(0,0,0)$.

To this system had trivial solutions the determinant of the system must be exactly equal to zero:

$$D=\begin{vmatrix}1 & 1 & 1\\ a & 4 & 1\\ 6 & a+2 & 2\end{vmatrix}$$

$$\begin{vmatrix}1 & 1 & 1\\ a & 4 & 1\\ 6 & a+2 & 2\end{vmatrix}\begin{matrix}1 & 1\\ a & 4\\ 6 & a+2\end{matrix} = 8+6+a(a+2)-2a-(a+2)-24 = 14+a^2+2a-2a-a-2-24 = a^2-a-12$$

$$a^2-a-12=0 \rightarrow a_{1,2}=\frac{1\pm\sqrt{49}}{2}=\frac{1\pm 7}{2} \rightarrow a_1=4; a_2=-3$$

Now we must examine the solutions for **both** to see what is happening. We return this value in the initial system:

for $a=4$

$$x+y+z=0$$
$$4x+4y+z=0$$
$$6x+(4+2)y+2z=0$$

$$x+y+z=0$$
$$4x+4y+z=0$$
$$6x+6y+2z=0 \ldots\ldots / :2$$

$$x+y+z=0$$
$$4x+4y+z=0$$
$$3x+3y+z=0$$

$$II-III \rightarrow x+y=0 \rightarrow \boxed{y=-x} \rightarrow x+y+z=0 \rightarrow \boxed{z=0}$$

Solutions are : $(x,y,z)=(x,-x,0) \quad x\in R$

for $a=-3$

$$x+y+z=0$$
$$-3x+4y+z=0$$
$$6x+(-3+2)y+2z=0$$

$$x+y+z=0$$
$$-3x+4y+z=0$$
$$6x-y+2z=0$$

$$III+I \rightarrow 7x+3z=0 \rightarrow \boxed{x=\frac{-3z}{7}}$$

$$x+y+z=0 \rightarrow \frac{-3z}{7}+y+z=0 \rightarrow y=-z+\frac{3z}{7} \rightarrow \boxed{y=\frac{-4z}{7}}$$

Solutios are : $(x,y,z)=(\frac{-3z}{7},\frac{-4z}{7},z) \quad z\in \mathrm{R}$

4. **Depending on the parameter *m*, discuss and solve the system:**

$$\begin{aligned} x+y+mz+t&=0\\ x-y-z-t&=0\\ mx+y+5z+3t&=0\\ x+5y+11z+8t&=0 \end{aligned}$$

Solution:

$$D=\begin{vmatrix} 1 & 1 & m & 1\\ 1 & -1 & -1 & -1\\ m & 1 & 5 & 3\\ 1 & 5 & 11 & 8 \end{vmatrix}$$

Properties of determinants will help us to help solve this determinant.

Do not chase and immediately try to make zero working with forms, sometimes it is easier to work with the columns .

$$D=\begin{vmatrix} 1 & 1 & m & 1\\ 1 & -1 & -1 & -1\\ m & 1 & 5 & 3\\ 1 & 5 & 11 & 8 \end{vmatrix} \sim \begin{matrix} II\text{column}+I\text{column}\rightarrow II\text{column}\\ III\text{column}+I\text{column}\rightarrow III\text{column}\\ IV\text{column}+I\text{column}\rightarrow IV\text{column}\end{matrix} = \sim \begin{vmatrix} 1 & 2 & m+1 & 2\\ 1 & 0 & 0 & 0\\ m & m+1 & m+5 & m+3\\ 1 & 6 & 12 & 9 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & m+1 & 2\\ \overset{-}{1} & \overset{+}{0} & \overset{-}{0} & \overset{+}{0}\\ m & m+1 & m+5 & m+3\\ 1 & 6 & 12 & 9 \end{vmatrix} = -1\begin{vmatrix} 2 & m+1 & 2\\ m+1 & m+5 & m+3\\ 6 & 12 & 9 \end{vmatrix}$$

$$-1\begin{vmatrix} 2 & m+1 & 2\\ m+1 & m+5 & m+3\\ 6 & 12 & 9 \end{vmatrix}=0\rightarrow \begin{vmatrix} 2 & m+1 & 2\\ m+1 & m+5 & m+3\\ 6 & 12 & 9 \end{vmatrix}=0$$

$$\begin{vmatrix} 2 & m+1 & 2\\ m+1 & m+5 & m+3\\ 6 & 12 & 9 \end{vmatrix}\begin{matrix} 2 & m+1\\ m+1 & m+5\\ 6 & 12 \end{matrix}=18(m+5)+6(m+1)(m+3)+24(m+1)-9(m+1)^2-24(m+3)-12(m+5)$$

$$=18m+90+6(m^2+4m+3)+\cancel{24m}+24-9(m^2+2m+1)\cancel{-24m}-72-12m-60$$
$$=\cancel{18m}+90+6m^2+24m+18+24-9m^2\cancel{-18m}-9-72-12m-60$$
$$=-3m^2+12m-9$$
$$D=0\rightarrow -3m^2+12m-9=0......./:(-3)$$
$$m^2-4m+3=0\rightarrow \boxed{m_1=3;\quad m_2=1}$$

for $m=3$

$$x+y+3z+t=0$$
$$x-y-z-t=0$$
$$3x+y+5z+3t=0$$
$$x+5y+11z+8t=0$$
$$(x,y,z,t)=(-z,-2z,z,0) \quad z\in R$$

for $m=1$

$$x+y+z+t=0$$
$$x-y-z-t=0$$
$$x+y+5z+3t=0$$
$$x+5y+11z+8t=0$$
$$(x,y,z,t)=(0,y,y,-2y) \quad y\in R$$

5. Depending on the parameters *n* and *m*, discuss and solve the system:

$$x-my=m$$
$$x+ny=n$$

Solution:

$$x-my=m$$
$$x+ny=n$$

$$D=\begin{vmatrix}1 & -m\\ 1 & n\end{vmatrix}=n+m$$

$$D_x=\begin{vmatrix}m & -m\\ n & n\end{vmatrix}=mn+mn=2mn$$

$$D_y=\begin{vmatrix}1 & m\\ 1 & n\end{vmatrix}=n-m$$

$$D\neq 0\rightarrow n+m\neq 0\rightarrow n\neq -m$$

$$x=\frac{D_x}{D}=\frac{2mn}{n+m}\rightarrow \boxed{x=\frac{2mn}{n+m}}$$

$$y=\frac{D_y}{D}=\frac{n-m}{n+m}\rightarrow \boxed{y=\frac{n-m}{n+m}}$$

for $n=-m$

$$D=0$$
$$D_x=2mn=-2m^2$$
$$D_y=n-m=-2m$$

If the value of *m* exactly zero, the system will have infinitely many solutions, and if m is different from zero the system is impossible.

for $m=0 \rightarrow n=0$

Restore these values in the system:

$x - my = m$

$x + ny = n$

$x - 0 \cdot y = 0$

$x + 0 \cdot y = 0$

From here we conclude that x must be equal to zero, and y is a random number.

Solutions writes as : $(x, y) = (0, y) \quad y \in R$