Exponentiation

$$\underbrace{a * a * \ldots * a}_{n-times} = a^n$$

If $a \in R \wedge a \neq 0$ and $n \in N$ then:

By definition is:

1) $a^0 = 1 \rightarrow \text{example}: \; 5^0 = 1, \; (-3)^0 = 1, \; \left(\frac{4}{7}\right)^0 = 1$

2) $a^{-n} = \frac{1}{a^n} \rightarrow \text{example}: \; 3^{-2} = \frac{1}{3^2} = \frac{1}{9}, \; 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

More rules:

3) $a^m \cdot a^n = a^{m+n} \quad \rightarrow \text{example}: \; 3^2 \cdot 3^5 = 3^{2+5} = 3^7$

4) $a^m : a^n = a^{m-n} \quad \rightarrow \text{example}: \; 7^{10} : 7^6 = 7^{10-6} = 7^4$

5) $(a^m)^n = a^{m \cdot n} \quad \rightarrow \text{example}: \; (2^3)^5 = 2^{3 \cdot 5} = 2^{15}$

6) $(a \cdot b) = a^n \cdot b^n \quad \rightarrow \text{example}: \; (12 \cdot 11)^5 = 12^5 \cdot 11^5$

7) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad \rightarrow \text{example}: \; \left(\frac{7}{4}\right)^2 = \frac{7^2}{4^2}$

8) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \quad \rightarrow \text{example}: \; \left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}$

We should pay attention to the record: $(-5)^2 = (-5)(-5) = 25$, while $-5^2 = -5 \cdot 5 = -25$.

In general, is:

$$(-a)^{2k} = a^{2k}$$
$$(-a)^{2k+1} = -a^{2k+1}$$

EXAMPLES:

1) Simplify: $\dfrac{(2^7:2^5)\cdot 2^3}{2^4:2^2}$

$$\frac{(2^7:2^5)\cdot 2^3}{2^4:2^2}=\frac{2^{7-5}\cdot 2^3}{2^{4-2}}=\frac{2^2\cdot 2^3}{2^2}=\frac{2^{2+3}}{2^2}=\frac{2^5}{2^3}=2^{5-2}=2^3=8$$

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2) Simplify: $\dfrac{3^5\cdot 9^3}{27^2\cdot 3}$

$$\frac{3^5\cdot 9^3}{27^2\cdot 3}=\frac{3^5\cdot (3^2)^3}{(3^3)^2\cdot 3^1}=\frac{3^5\cdot \cancel{3^6}}{\cancel{3^6}\cdot 3^1}=\frac{3^5}{3^1}=3^{5-1}=3^4=3\cdot 3\cdot 3\cdot 3=81$$

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3) Simplify: $\dfrac{(x^4)^3\cdot x^3:x^5}{(x^5:x^2)^3}$

$$\frac{(x^4)^3\cdot x^3:x^5}{(x^5:x^2)^3}=\frac{x^{12}\cdot x^3:x^5}{(x^{5-2})^3}=\frac{x^{12+3-5}}{(x^3)^3}=\frac{x^{10}}{x^9}=x^{10-9}=x^1=x$$

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4) Simplify: $\dfrac{3^{n+1}\cdot 3^{n+2}}{3^{2n+4}}$

$$\frac{3^{n+1}\cdot 3^{n+2}}{3^{2n+4}}=\frac{3^{n+1+n+2}}{3^{2n+4}}=\frac{3^{2n+3}}{3^{2n+4}}=$$
$$=3^{(2n+3)-(2n+4)}=3^{2n+3-2n-4}=3^{-1}=\frac{1}{3^1}=\frac{1}{3}$$

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5) Simplify : $0{,}5^{-1}+0{,}25^{-2}+0{,}125^{-3}+0{,}0625^{-4}$

$$0{,}5^{-1}+0{,}25^{-2}+0{,}125^{-3}+0{,}0625^{-4}=$$
$$\left(\frac{1}{2}\right)^{-1}+\left(\frac{1}{4}\right)^{-2}+\left(\frac{1}{8}\right)^{-3}+\left(\frac{1}{16}\right)^{-4}=$$
$$\left(\frac{2}{1}\right)^{1}+\left(\frac{4}{2}\right)^{2}+\left(\frac{8}{1}\right)^{3}+\left(\frac{16}{1}\right)^{4}=$$
$$2^1+4^2+8^3+16^4=2+16+512+65536=66066$$

6) $1^{-1}+2^{-2}+3^{-3}+(-1)^{-1}+(-2)^{-2}+(-3)^{-3}=?$

$$1^{-1}+2^{-2}+3^{-3}+(-1)^{-1}+(-2)^{-2}+(-3)^{-3}=$$
$$\frac{1}{1}+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{(-1)^1}+\frac{1}{(-2)^2}+\frac{1}{(-3)^3}=$$
$$1+\frac{1}{4}+\frac{1}{27}-1+\frac{1}{4}-\frac{1}{27}=\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$$

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7) If $a=5^3\cdot\left(\frac{1}{4}\right)^{-4}\cdot\left(\frac{3}{2}\right)^2$ and $b=10^3\left(\frac{5}{3}\right)^{-2}$ find $a\cdot b^{-1}=?$

$$a=5^3\cdot\left(\frac{1}{4}\right)^{-4}\cdot\left(\frac{3}{2}\right)^2=5^3\cdot\left(\frac{4}{1}\right)^4\cdot\frac{3^2}{2^2}=\frac{5^3\cdot4^4\cdot3^2}{2^2}$$
$$=\frac{5^3\cdot(2^2)^4\cdot3^2}{2^2}=5^3\cdot(2^2)^3\cdot3^2=5^3\cdot2^6\cdot3^2$$
$$b=10^3\cdot\left(\frac{5}{3}\right)^{-2}=10^3\cdot\left(\frac{3}{5}\right)^2=\frac{10^3\cdot3^2}{5^2}=\frac{5^3\cdot2^3\cdot3^2}{5^2}=5\cdot2^3\cdot3^2$$

Finally: calculate $a\cdot b^{-1}$

$$a\cdot b^{-1}=5\cdot2^3\cdot3^2\cdot\frac{1}{5\cdot2^3\cdot3^2}=5^2\cdot2^3=25\cdot8=200$$

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8) $\left(\left(\frac{5x^{-5}}{2y^{-2}}\right)^{-2}\cdot\left(\frac{y^{-1}}{5x^{-1}}\right)^{-3}\right):10x^2y^{-3}=?$

$$\left(\left(\frac{5x^{-5}}{2y^{-2}}\right)^{-2}\cdot\left(\frac{y^{-1}}{5x^{-1}}\right)^{-3}\right):10x^2y^{-3}=$$
$$\left(\frac{5^{-2}\cdot x^{10}}{2^{-2}\cdot y^4}\cdot\frac{y^3}{5^{-3}\cdot x^3}\right):10x^2y^{-3}=$$
$$(5^{-2+3}\cdot x^{10-3}\cdot y^{3-4}\cdot2^2):10x^2y^{-3}=$$
$$(5^1\cdot x^7\cdot y^{-1}\cdot4):10x^2y^{-3}=$$
$$\frac{20}{10}x^{7-2}y^{-1-(-3)}=2x^5y^{-1+3}=2x^5y^2$$

9) If $10^x = \dfrac{\frac{1}{2}10^{-3} + \frac{1}{2}10^{-4}}{55 \cdot 10^{-7}}$ **Find x=?**

$$\frac{\frac{1}{2}10^{-3} + \frac{1}{2}10^{-4}}{55 \cdot 10^{-7}} = \frac{\frac{1}{2000} + \frac{1}{20000}}{\frac{55}{10000000}} =$$

$$\frac{\frac{1}{2000}\left(1 + \frac{1}{10}\right)}{\frac{55}{10000000}} = \frac{\frac{1}{2000} \cdot \frac{11}{10}}{\frac{55}{10000000}} = \frac{\frac{11}{20000}}{\frac{55}{10000000}} =$$

$$\frac{11 \cdot 10000000}{20000 \cdot 55} = \frac{11 \cdot 10000000}{11 \cdot 100000} = \frac{10000000}{100000} = 100 = 10^2$$

Now $10^x = 10^2$, then $x = 2$

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10)

a) $A \cdot 10^{-5} = 0{,}2 \cdot 0{,}008$ **b)** $B \cdot 10^{-6} = 0{,}04 \cdot 0{,}006$

A = ?

$$A \cdot 10^{-5} = 0{,}2 \cdot 0{,}008$$
$$A \cdot 10^{-5} = 2 \cdot 10^{-1} \cdot 8 \cdot 10^{-3}$$
$$A \cdot 10^{-5} = 16 \cdot 10^{-4}$$
$$A = \frac{16 \cdot 10^{-4}}{10^{-5}}$$
$$A = 16 \cdot 10^{-4-(-5)}$$
$$A = 16 \cdot 10^{-4+5}$$
$$A = 16 \cdot 10$$
$$A = 160$$

B=?

$$B \cdot 10^{-6} = 0{,}04 \cdot 0{,}006$$
$$B \cdot 10^{-6} = 4 \cdot 10^{-2} \cdot 6 \cdot 10^{-3}$$
$$B \cdot 10^{-6} = 24 \cdot 10^{-5}$$
$$B = \frac{24 \cdot 10^{-5}}{10^{-6}}$$
$$B = 24 \cdot 10^{-5+6}$$
$$B = 24 \cdot 10$$
$$B = 240$$

Here we use a real number in the recording system with the 10.
This is a good options when the number is very big.

For example:

1) speed of light is approximately $c = 300000000 m/s$, “easier"notes is $c \approx 3 \cdot 10^8 m/s$

2) $\dfrac{1}{500000} = \dfrac{1}{5 \cdot 10^5} = \dfrac{1}{5} \cdot 10^{-5} = \dfrac{2}{10} \cdot 10^{-5} = 2 \cdot 10^{-1} \cdot 10^{-5} = 2 \cdot 10^{-6}$

3) $0{,}000069 = 6{,}9 \cdot 10^{-5} \approx 7 \cdot 10^{-5}$

4) The surface of the earth is $510083000 km^2$ but we note: $\approx 5 \cdot 10^8 km^2$

11) $\left(\frac{3a^{-x}}{1-a^{-x}}-\frac{2a^{-x}}{1+a^{-x}}-\frac{a^{x}}{a^{2x}-1}\right):\frac{a^{-x}}{a^{x}-a^{-x}}=?$

$$\left(\frac{3a^{-x}}{1-a^{-x}}-\frac{2a^{-x}}{1+a^{-x}}-\frac{a^{x}}{a^{2x}-1}\right):\frac{a^{-x}}{a^{x}-a^{-x}}=$$

$$\left(\frac{\frac{3}{a^{x}}}{1-\frac{1}{a^{x}}}-\frac{\frac{2}{a^{x}}}{1+\frac{1}{a^{x}}}-\frac{a^{x}}{a^{2x}-1}\right):\frac{\frac{1}{a^{x}}}{a^{x}-\frac{1}{a^{x}}}=$$

$$\left(\frac{\frac{3}{a^{x}}}{\frac{a^{x}-1}{a^{x}}}-\frac{\frac{2}{a^{x}}}{\frac{a^{x}+1}{a^{x}}}-\frac{a^{x}}{a^{2x}-1}\right):\frac{\frac{1}{a^{x}}}{\frac{a^{2x}-1}{a^{x}}}=$$

$$\left(\frac{3}{a^{x}-1}-\frac{2}{a^{x}+1}-\frac{a^{x}}{(a^{x}-1)(a^{x}+1)}\right):\frac{1}{a^{2x}-1}=$$

$$\frac{3(a^{x}+1)-2(a^{x}-1)-a^{x}}{(a^{x}-1)(a^{x}+1)}\cdot\frac{a^{2x}-1}{1}=$$

$$\frac{3a^{x}+3-2a^{x}+2-a^{x}}{\cancel{(a^{x}-1)(a^{x}+1)}}\cdot\frac{\cancel{(a^{x}-1)(a^{x}+1)}}{1}=$$

$$=3+2=5$$

12) $\left(\frac{x-x^{-2}}{x^{-2}+x^{-1}+1}-\frac{x-x^{-1}}{1+x^{-2}+2\cdot x^{-1}}\right):\frac{1-x^{-1}}{1+x^{-1}}=?$

$$\left(\frac{x-x^{-2}}{x^{-2}+x^{-1}+1}-\frac{x-x^{-1}}{1+x^{-2}+2\cdot x^{-1}}\right):\frac{1-x^{-1}}{1+x^{-1}}=$$

$$\left(\frac{x-\frac{1}{x^{2}}}{\frac{1}{x^{2}}+\frac{1}{x}+1}-\frac{x-\frac{1}{x}}{1+\frac{1}{x^{2}}+\frac{2}{x}}\right):\frac{1-\frac{1}{x}}{1+\frac{1}{x}}=$$

$$\left(\frac{\frac{x^{3}-1}{x^{2}}}{\frac{1+x+x^{2}}{x^{2}}}-\frac{\frac{x^{2}-1}{x}}{\frac{x^{2}+1+2x}{x^{2}}}\right):\frac{\frac{x-1}{x}}{\frac{x+1}{x}}=$$

$$\left(\frac{(x-1)\,\cancel{(x^{2}+x+1)}}{\cancel{x^{2}+x+1}}-\frac{x(x-1)\,\cancel{(x+1)}}{(x+1)^{\cancel{2}}}\right):\frac{x-1}{x+1}=$$

$$\left(\frac{x-1}{1}-\frac{x(x-1)}{x+1}\right):\frac{x-1}{x+1}=$$

$$\frac{(x-1)(x+1)-x(x-1)}{x+1}\cdot\frac{x+1}{x-1}=$$

$$\frac{\cancel{(x-1)}\left[(x+1)-x\right]}{\cancel{x+1}}\cdot\frac{\cancel{x+1}}{\cancel{x-1}}=x+1-x=1$$

13) $A=\left(\frac{a^{n}}{1-a^{-n}}+\frac{a^{-n}}{1+a^{-n}}\right)-\left(\frac{a^{n}}{1+a^{-n}}+\frac{a^{-n}}{1-a^{-n}}\right)=?$

$$A=\left(\frac{a^{n}}{1-a^{-n}}+\frac{a^{-n}}{1+a^{-n}}\right)-\left(\frac{a^{n}}{1+a^{-n}}+\frac{a^{-n}}{1-a^{-n}}\right)$$

$$A=\left(\frac{a^{n}}{1-\frac{1}{a^{n}}}+\frac{\frac{1}{a^{n}}}{1+\frac{1}{a^{n}}}\right)-\left(\frac{a^{n}}{1+\frac{1}{a^{n}}}+\frac{\frac{1}{a^{n}}}{1-\frac{1}{a^{n}}}\right)$$

$$A=\left(\frac{\frac{a^{n}}{1}}{\frac{a^{n}-1}{a^{n}}}+\frac{\frac{1}{a^{n}}}{\frac{a^{n}+1}{a^{n}}}\right)-\left(\frac{a^{n}}{\frac{a^{n}+1}{a^{n}}}+\frac{\frac{1}{a^{n}}}{\frac{a^{n}-1}{a^{n}}}\right)$$

$$A=\left(\frac{a^{2n}}{a^{n}-1}+\frac{1}{a^{n}+1}\right)-\left(\frac{a^{2n}}{a^{n}+1}+\frac{1}{a^{n}-1}\right)$$

$$A=\frac{a^{2n}(a^{n}+1)+1(a^{n}-1)}{(a^{n}-1)(a^{n}+1)}-\frac{a^{2n}(a^{n}-1)+1(a^{n}+1)}{(a^{n}+1)(a^{n}-1)}$$

$$A=\frac{a^{3n}+a^{2n}+a^{n}-1-(a^{3n}-a^{2n}+a^{n}+1)}{(a^{n}-1)(a^{n}+1)}$$

$$A=\frac{a^{3n}+a^{2n}+a^{n}-1-a^{3n}+a^{2n}-a^{n}-1}{(a^{n}-1)(a^{n}+1)}$$

$$A=\frac{2a^{2n}-2}{(a^{n}-1)(a^{n}+1)}=\frac{2(a^{2n}-1)}{(a^{n}-1)(a^{n}+1)}=\frac{2\,(a^{n}-1)\,(a^{n}+1)}{(a^{n}-1)\,(a^{n}+1)}=2$$