Vectors in space (Part II)

Mixed product of three vectors

Mixed product is $(\vec{a}\times\vec{b})\circ\vec{c}$. Usually marked with $\left[\vec{a},\vec{b},\vec{c}\right]$ So: $(\vec{a}\times\vec{b})\circ\vec{c} = \left[\vec{a},\vec{b},\vec{c}\right]$

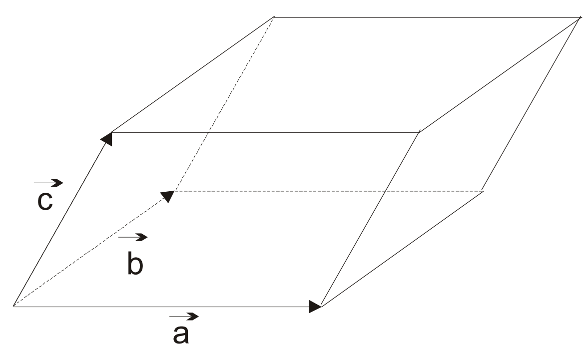
How it calculated?

If given vectors are: $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$ and $\vec{c}=(c_1,c_2,c_3)$ then:

$$(\vec{a}\times\vec{b})\circ\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

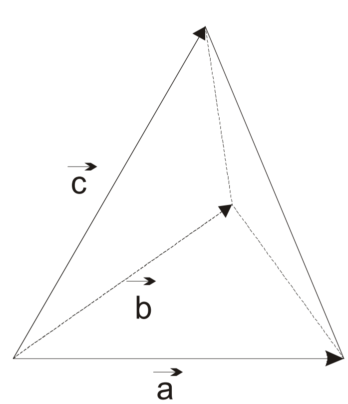
For what is it used?

i) Absolute value of mixed products three vectors is the same as the volume parallelepiped **constructed over them, that is:** $V(\vec{a},\vec{b},\vec{c}) = \left|(\vec{a}\times\vec{b})\circ\vec{c}\right|$



ii) **Volume of the pyramid (tetrahedra) constructed over vectors a, b, c, is:**

$$V = \frac{1}{6}\left|(\vec{a}\times\vec{b})\circ\vec{c}\right|$$



Why $\frac{1}{6}$ in the formula?

Since that from earlier we know the volume of the pyramid is :

$$V = \frac{1}{3} B\, H$$

As the base is triangle, B = $\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$ And then:

$$V = \frac{1}{3} B\, H = \frac{1}{3}\,\frac{1}{2}\left|\vec{a}\times\vec{b}\right| H = \frac{1}{6}\left|(\vec{a}\times\vec{b})\circ\vec{c}\right|$$

How to find the height H of the pyramid?

$\frac{1}{6}\left|(\vec{a}\times\vec{b})\circ\vec{c}\right| = \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, Then find the base B= $\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$ and replace in H = $\frac{3V}{B}$.

iii) Vectors lie in a plane

Vectors $\vec{a},\vec{b},\vec{c}$ lie in a plane if and only if their mixed product is equal to zero.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Examples:

1. **Calculate volume parallelepiped constructed over vectors:** $\vec{a}(0,1,1), \vec{b}(1,0,1), \vec{c}(1,1,0)$

Solution:

$$V(\vec{a},\vec{b},\vec{c}) = \left|(\vec{a}\times\vec{b})\circ\vec{c}\right| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} =$$

$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}\begin{matrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{matrix} = (0 + 1 + 1) - (0+0+0) = 2$ [see determinants]

Therefore, V = 2

2. Given vectors are vectors $\vec{a}(\ln(p-2),-2,6),\vec{b}(p,-2,5),\vec{c}(0,-1,3)$. Determine the real number *p*, for which vectors lie in a plane .

Solution:

As we said, must be $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$\begin{vmatrix} \ln(p-2) & -2 & 6 \\ p & -2 & 5 \\ 0 & -1 & 3 \end{vmatrix}$ = Develop is the first column= ln(p-2)[-6+5] – p [-6+6] = - ln(p-2)

Must be - ln (P-2) = 0 [See file logarithms]

p - 2 = 1, and **p = 3 is requested solution.**

3. Given vectors are $\vec{a}(1,1,-1),\vec{b}(-2,-1,2),\vec{c}(1,-1,2)$,

Set apart vector $\vec{c}$ in directions of vectors $\vec{a},\vec{b}$ and $\vec{a}\times\vec{b}$

Solution: First, we find $\vec{a}\times\vec{b}$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -2 & -1 & 2 \end{vmatrix} = 1\vec{i} - 0\vec{j} + 1\vec{k} = (1,0,1)$$

$\vec{c}$ = m$\vec{a}$ + n$\vec{b}$ + p ($\vec{a}\times\vec{b}$) where m,n and p are constants that must be found.

(1,-1,2) = m (1,1,-1) + n (-2,-1,2) + p (1,0,1) crossing in the system of equations:

$$\left.\begin{aligned} 1 &= 1m - 2n + 1p \\ -1 &= 1m - 1n + 0p \\ 2 &= -1m + 2n + 1p \end{aligned}\right\} \quad \begin{aligned} m - 2n + p &= 1 \\ m - n &= -1 \\ -m + 2n + p &= 2 \end{aligned}$$

gather first and third ... and p=$\frac{3}{2}$

Return p = $\frac{3}{2}$ in other two equations and obtain: m = $-\frac{3}{2}$ and n = $-\frac{1}{2}$

Let's go back now:

$$\vec{c} = m\vec{a} + n\vec{b} + p\,(\vec{a}\times\vec{b})$$

$\vec{c} = -\frac{3}{2}\,\vec{a} - \frac{1}{2}\,\vec{b} + \frac{3}{2}\,(\vec{a}\times\vec{b})$ **is the final solution**

4. Given vectors are $\vec{a}(m-1,1,1), \vec{b}(-1, m+1, 0), \vec{c}(m,2,1)$. Determine the value of parameter m so that vectors $\vec{a}, \vec{b}, \vec{c}$ lie in a plane and decompose $\vec{a}$ in components by other two.

Solution:

$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$, it is $\begin{vmatrix} m-1 & 1 & 1 \\ -1 & m+1 & 0 \\ m & 2 & 1 \end{vmatrix} = 0$ this determined develop by the third column ...=

= -2 – m(m+1) + (m-1)(m+1) +1 = 0

= - 2 – m² – m + m² – 1 +1 = 0   So: **m = - 2**

When to replace m = -2:

$$\vec{a} = (-3,1,1)$$
$$\vec{b} = (-1,-1,0)$$
$$\vec{c} = (-2,2,1)$$

Go to decompose :

$\vec{a}$ **= m** $\vec{b}$ + **n** $\vec{c}$

$(-3,1,1)$ **= m** $(-1,-1,0)$ + **n** $(-2,2,1)$ crossing in the system of equations

-3 = - m – 2n

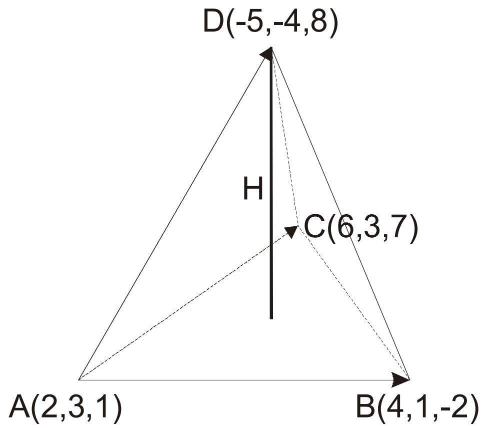
1 = - m + 2n

1 = 0m + n ⟶ here is n = 1 and to change in above two equations...m = 1

So $\vec{a}$ **= m** $\vec{b}$ + **n** $\vec{c}$ **and** $\vec{a}$ = $\vec{b}$ + $\vec{c}$ **is final solution**

5. We know that the vertices of a tetrahedron are A (2,3,1), B (4.1, -2), C (6,3,7) and D (-5, -4.8). Determine volume tetrahedra and height devolved from the vertice D on the ABC.

Solution:
First, we draw picture and post the problem:



Create vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$

$\overrightarrow{AB}$ = (4-2,1-3,-2-1) = (2, -2, -3)

$\overrightarrow{AC}$ = (6-2,3-3,7-1) = (4, 0, 6)

$\overrightarrow{AD}$ = (-5-2,-4-3,8-1) = (-7,-7,7)

Volume tetrahedra can be found by the formula: $\frac{1}{6}\left|(\vec{a}\times\vec{b})\circ\vec{c}\right| = \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$

$$\frac{1}{6}\begin{vmatrix} 2 & -2 & -3 \\ 4 & 0 & 6 \\ -7 & -7 & 7 \end{vmatrix} = \frac{308}{6}$$

Still looking for area ABC: B=$\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -3 \\ 4 & 0 & 6 \end{vmatrix} = -12\,\vec{i} + 24\,\vec{j} + 8\,\vec{k} = (-12, 24, 8)$$

$$B = \frac{1}{2}\sqrt{(-12)^2 + 24^2 + 8^2} = 14$$

Use the $H = \frac{3V}{B}$.

$$H = \frac{3\frac{308}{6}}{14} = 11$$ Therefore, the required height is H = 11

Note:

If you seek a different height, for example, from vertice C , is analogous.

Volume find, then area $\overrightarrow{AB} \times \overrightarrow{AD}$ and replace in $H = \frac{3V}{B}$.