Functional equation

Solving process:

i) "It", which is in brackets replace with t (replacement)
ii) From there, express x
iii) Back in the starting equation, f (t) =... and where you see x replace it with what we have expressed
iv) Simplify that term, who is now all "by t" and **replace t with x**

EXAMPLES:

1) Solve the functional equation: $f(x+1) = x^2 - 3x + 2$

Solution:

$f(x+1) = x^2 - 3x + 2$ "It", which is in brackets replace with t

$x + 1 = t$ From there, express x

$x = t - 1$ Back in the starting equation, f (t) =...

$f(t) = (t-1)^2 - 3(t-1) + 2$

$f(t) = t^2 - 2t + 1 - 3t + 3 + 2$ Simplify that term, who is now all "by t"

$f(t) = t^2 - 5t + 6$ **replace t with x**

$f(x) = x^2 - 5x + 6$ $\longrightarrow$ final solution of the functional equation.

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2) Solve the functional equation: $f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2}$

Solution:

$$f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2}$$

$$\frac{1}{x} = t$$

$$f(t) = \frac{1}{t} + \sqrt{1 + \frac{1}{t^2}}$$

$f(t) = \frac{1}{t} + \sqrt{\frac{t^2+1}{t^2}} \longrightarrow f(t) = \frac{1}{t} + \frac{\sqrt{t^2+1}}{t}$ **replace t with x** $f(x) = \frac{1+\sqrt{x^2+1}}{x}$ is final solution

3) Solve the functional equation: $f(\frac{x}{x+1}) = x^2$

Solution:

$f(\frac{x}{x+1}) = x^2$

$\frac{x}{x+1} = t$

x = t (x+1)

x = t x + t

x – tx = t

x (1 – t) = t

$x = \frac{t}{1-t}$

$f(\frac{x}{x+1}) = x^2$

$f(t) = (\frac{t}{1-t})^2$ **replace t with x** $f(x) = (\frac{x}{1-x})^2$ is final solution

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4) Solve the functional equation: $f(\frac{x+2}{2x+1}) = 5x+3$

Solution:

$f(\frac{x+2}{2x+1}) = 5x+3$

$\frac{x+2}{2x+1} = t$

x + 2 = t (2x + 1)

x + 2 = 2tx + t

x – 2tx = t – 2

x (1 – 2t) = t – 2

$x = \frac{t-2}{1-2t}$ ⟶ $f(\frac{x+2}{2x+1}) = 5x+3$

f (t) = 5 $\frac{t-2}{1-2t}$ **+ 3**

f (t) = $\frac{5t-10}{1-2t} + \frac{3(1-2t)}{1-2t} = \frac{5t-10+3-6t}{1-2t} = \frac{-t-7}{1-2t}$

f (t) = $\frac{t+7}{2t-1}$

f (x) = $\frac{x+7}{2x-1}$ is final solution.

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5) If $f(\frac{x}{x+1}) = (x-1)^2$ **calculate f(3).**

Solution:

First, we must find f (x).

$f(\frac{x}{x+1}) = (x-1)^2$

$\frac{x}{x+1} = t$

x = t (x+1)

x = t x + t

x – tx = t

x (1 – t) = t

$x = \frac{t}{1-t}$ ⟶ $f(\frac{x}{x+1}) = (x-1)^2$

f (t) = ($\frac{t}{1-t}$ **- 1)^2** **Now, t replace with 3, because we search f (3) ...**

f (3) = ($\frac{3}{1-3}$ **- 1)^2 =** $\frac{25}{4}$

6) Solve the functional equation: $f(x+\frac{1}{x})=x^2+\frac{1}{x^2}$

Solution:

$f(x+\frac{1}{x})=x^2+\frac{1}{x^2}$ replacement $x+\frac{1}{x}=$ t , if you try from here to express x, as it should, **there is a problem….**

$x+\frac{1}{x}=$ t all multiply with x ..

$x^2+1=$ xt

x^2- xt + 1 = 0 this is a square equation but does not give us a "good" solution

What should you do?

Must use a different method:

$x+\frac{1}{x}=$ t

$(x+\frac{1}{x})^2=t^2$

$x^2+2x\frac{1}{x}+\frac{1}{x^2}=t^2$

$x^2+2+\frac{1}{x^2}=t^2$

$x^2+\frac{1}{x^2}=t^2-2$ now back in the equation $f(x+\frac{1}{x})=x^2+\frac{1}{x^2}$

$f(x+\frac{1}{x})=x^2+\frac{1}{x^2}$ ⟶ **$f(t)=t^2-2$ then $f(x)=x^2-2$ is final solution**

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7) Solve the functional equation: $f\left(\frac{x+1}{x-2}\right)+2f\left(\frac{x-2}{x+1}\right)=x$

Solution:

This task can not do "classic" but must use a different method:

If we have replacement $\frac{x-2}{x+1}=t$ **then** $\frac{x+1}{x-2}=\frac{1}{t}$ **and from here we have:**

x-2 = t (x +1) ⟶ x – 2= tx + t ⟶ x – tx = t + 2 ⟶ x (1-t)= t + 2 ⟶ $x=\frac{t+2}{1-t}$

Back in the equation:

$$f\left(\frac{x+1}{x-2}\right)+2f\left(\frac{x-2}{x+1}\right)=x$$

$f(\frac{1}{t})+2f(t)=\frac{t+2}{1-t}$ We have received new equation **, trick is to place** $\frac{1}{t}$ **instead of t.**

$$f(t)+2f(\frac{1}{t})=\frac{\frac{1}{t}+2}{1-\frac{1}{t}}=\frac{\frac{1+2t}{t}}{\frac{t-1}{t}}=\frac{1+2t}{t-1}$$

Now make the system of this two equations:

$$f(\frac{1}{t})+2f(t)=\frac{t+2}{1-t}$$
$$2f(\frac{1}{t})+f(t)=\frac{1+2t}{t-1}$$

Multiply the first equation with -2 and gather these two equations ...

$$-4f(t)-2f(\frac{1}{t})=-2\frac{t+2}{1-t}$$
$$f(t)+2f(\frac{1}{t})=\frac{1+2t}{t-1}$$

$$-3f(t)=\frac{-2t-4}{1-t}+\frac{1+2t}{t-1}=\frac{2t+4}{t-1}+\frac{1+2t}{t-1}=\frac{4t+5}{t-1}$$

So: $-3f(t)=\frac{4t+5}{t-1}$

$f(t)=\frac{4t+5}{-3(t-1)}$ ⟶ $f(t)=\frac{4t+5}{3-3t}$ **replace t with x**

$f(x)=\frac{4x+5}{3-3x}$ **is final solution**