Trigonometric inequalities

When we seeking solutions of inequalities, we first solve the appropriate equation, and then find intervals that meet the inequalities.

Inequalities sinx > *a* and sinx < *a*

$\sin x > a$
- $a < -1$ -every number is solution, $\forall x \in R$
- $-1 \le a \le 1$ - we must solve
- $a \ge 1$ -no solution

$\sin x < a$
- $a \le -1$ - no solution
- $-1 \le a \le 1$ - we must solve
- $a > 1$ - every number is solution, $\forall x \in R$

Example 1. Solve the inequalities:

a) $\sin x > -2$

b) $\sin x > \frac{1}{2}$

c) $\sin x > 3$

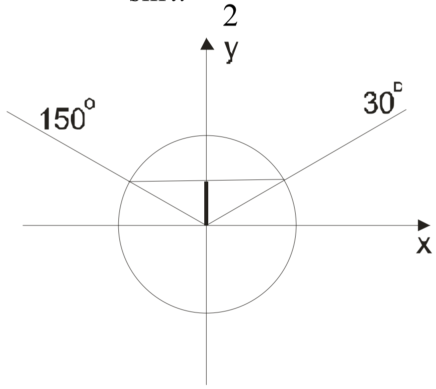
Solution:

a) $\sin x > -2$ because $-1 \le \sin x \le 1$ every $x \in R$ is solution.

b) $\sin x > \frac{1}{2}$

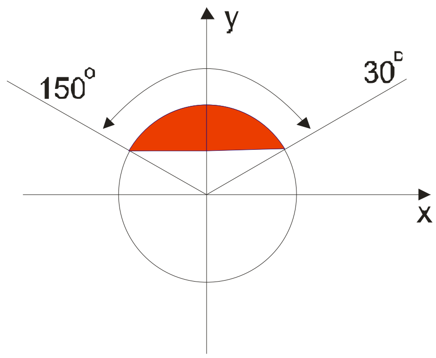
First, we solve the appropriate equation:

$\sin x = \frac{1}{2}$



Therefore, equations solutions are:
$$x = \frac{\pi}{6} + 2k\pi$$
$$x = \frac{5\pi}{6} + 2k\pi$$

Now, think! Since we need to be $\sin x > \frac{1}{2}$, we take the "upper part".



So: $\frac{\pi}{6} < x < \frac{5\pi}{6}$

We must add periodicity:

$\frac{\pi}{6} + 2k\pi < x < \frac{5\pi}{6} + 2k\pi$, $k \in Z$ is solution!

c) $\sin x > 3$

This is impossible, therefore, inequalities has no solution!

Example 2. Solve the inequalities:

a) $\sin x < -2$

b) $\sin x \leq -\frac{\sqrt{2}}{2}$

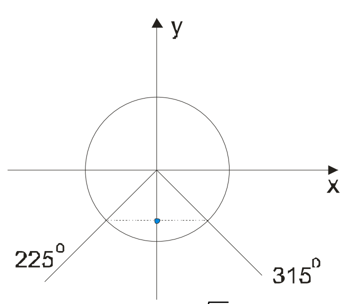
c) $\sin x < 5$

Solution:

a) $\sin x < -2 \Rightarrow$ How is $-1 \leq \sin x \leq 1$, therefore never be less than -2, the inequalities has no solution.

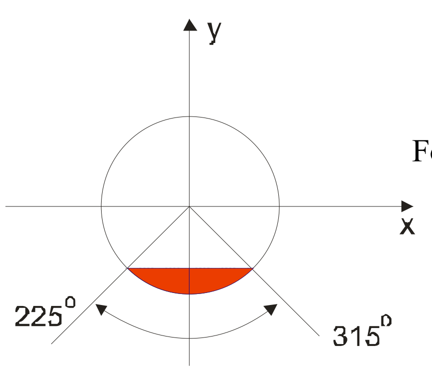
b) $\sin x \leq -\frac{\sqrt{2}}{2}$

First, we solve solution: $\sin x = -\frac{\sqrt{2}}{2}$



$x = \frac{5\pi}{4} + 2k\pi$

$x = \frac{7\pi}{4} + 2k\pi$



For inequalities $\sin x \leq -\frac{\sqrt{2}}{2}$ we need "lower" part! So: $\frac{5\pi}{4} \leq x \leq \frac{7\pi}{4}$

$\frac{5\pi}{4} + 2k\pi \leq x \leq \frac{7\pi}{4} + 2k\pi, k \in Z$

c) $\sin x < 5$

How is $-1 \le \sin x \le 1$, these inequalities are always satisfied, $\forall x \in R$ is solution.

Inequalities cosx > b and cosx < b

$$\cos x > b \quad \begin{cases} b < -1 \text{ - every number is solution, } \forall x \in R \\ -1 \le b \le 1 \text{ - we must solve} \\ b \ge 1 \text{ - no solution} \end{cases}$$

$$\cos x < b \quad \begin{cases} b < -1 \text{ - no solution} \\ -1 \le b \le 1 \text{ - we must solve} \\ b > 1 \text{ - every number is solution, } \forall x \in R \end{cases}$$

Example 1. Solve the inequalities:

a) $\cos x > -2$

b) $\cos x > \frac{1}{2}$

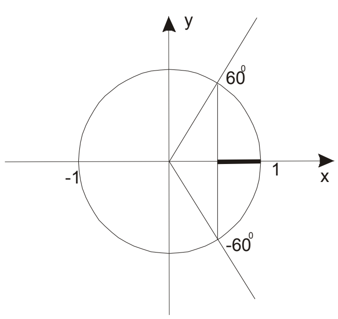
c) $\cos x > \frac{3}{2}$

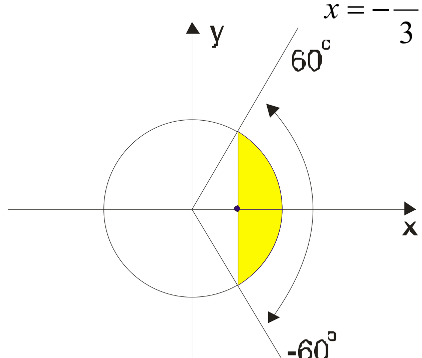
Solution:

a) $\cos x > -2$ every number is solution, $\forall x \in R$

b) $\cos x > \frac{1}{2}$

First, we solve $\cos x = \frac{1}{2}$ ⟶ $x = \frac{\pi}{3} + 2k\pi$, $x = -\frac{\pi}{3} + 2k\pi$





For the solutions we need angles which is the cosine of more than $\frac{1}{2}$,that means "right".

So: $-\frac{\pi}{3} + 2k\pi < x < \frac{\pi}{3} + 2k\pi \quad , \quad k \in Z$

c) $\cos x > \frac{3}{2}$

The inequalities has no solutions, because the largest value for the "cosine", as we know, can be 1.

Example 2. Solve the inequalities:

a) $\cos x < -2$

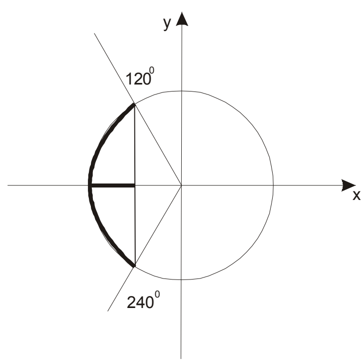
b) $\cos x \le -\frac{1}{2}$

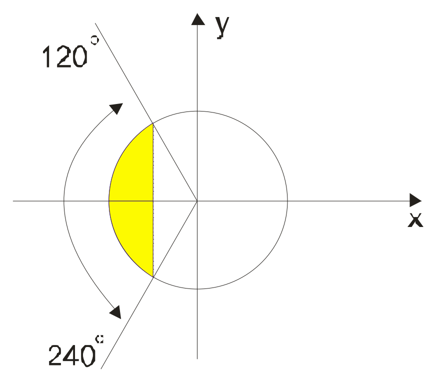
c) $\cos x < 2$

Solution:

a) $\cos x < -2$ - no solution.

b) $\cos x \le -\frac{1}{2} \longrightarrow \cos x = -\frac{1}{2}$





$x = \frac{2\pi}{3} + 2k\pi$

$x = \frac{4\pi}{3} + 2k\pi$

For solution of inequalities $\cos x \le -\frac{1}{2}$ we need "left" part!

Solution is: $\frac{2\pi}{3} + 2k\pi \le x \le \frac{4\pi}{3} + 2k\pi$

c) $\cos x < 2 \longrightarrow$ every number is solution, $\forall x \in R$

Inequalities with tgx and ctgx:

These inequalities as opposed to those with sinx and cosx always have the solutions and take value from the whole set R.

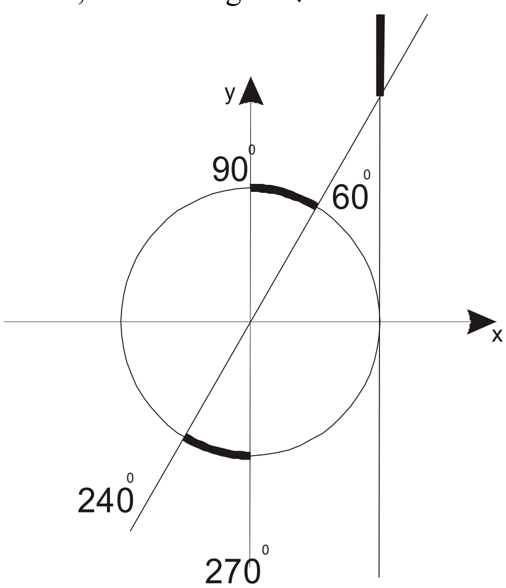
Example 1. Solve the inequalities:

a) $tgx > \sqrt{3}$
b) $tgx < 1$

Solution:

$tgx = \sqrt{3} \longrightarrow x = 60^o + k\pi$

Think, where is $tgx > \sqrt{3}$?



First interval is make by angles from 60^o to 90^o.

Second interval from 240^o to 270^o.

So here we have two intervals with solutions!

$60^o < x < 90^o$ and $240^o < x < 270^o$ Add period…

$\frac{\pi}{3} + k\pi < x < \frac{\pi}{2} + k\pi$, $k \in Z$

$\frac{4\pi}{3} + k\pi < x < \frac{3\pi}{2} + k\pi$, $k \in Z$ or, we can write:

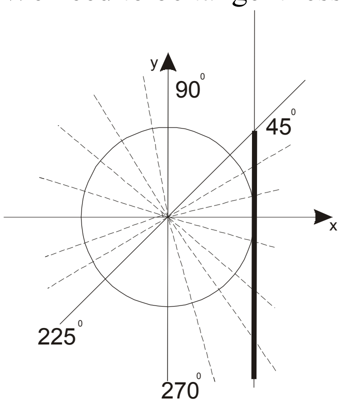
$x \in (\frac{\pi}{3} + k\pi, \frac{\pi}{2} + k\pi)$, $k \in Z$

$x \in (\frac{4\pi}{3} + k\pi, \frac{3\pi}{2} + k\pi)$

b) $tgx < 1$

$tgx = 1$, Solutions are angles 45^o and 225^o.

We need to be tangent less than 1 (bold)



Again we have two solutions!

$$-\frac{\pi}{2} < x < \frac{\pi}{4} \quad \text{and} \quad \frac{\pi}{2} < x < \frac{5\pi}{4} \quad \text{or, we can write:}$$

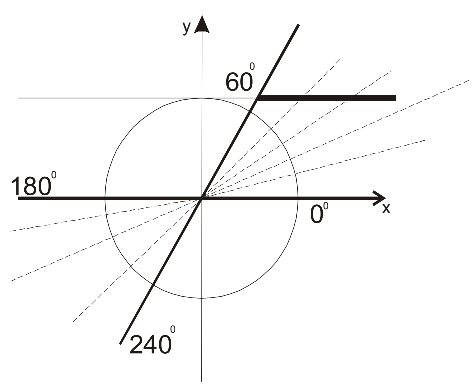
$$x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{4} + k\pi) \cup (\frac{\pi}{2} + k\pi, \frac{5\pi}{4} + k\pi)$$
$$k \in Z$$

Example 2. Solve the inequalities:

a) $ctgx > \frac{\sqrt{3}}{3}$

b) $ctgx < 0$

Solution:

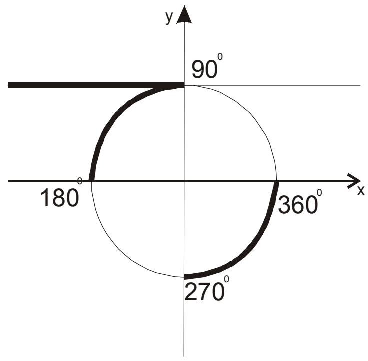


a) $ctgx = \frac{\sqrt{3}}{3} \Rightarrow x = 60^o$ and $x = 240^o$

Again two intervals: $0 < x < \frac{\pi}{3}$ and $\pi < x < \frac{4\pi}{3}$

Solution is: $x \in (0 + k\pi, \frac{\pi}{3} + k\pi) \cup (\pi + k\pi, \frac{4\pi}{3} + k\pi)$, $k \in Z$

b) $ctgx = 0$



$\frac{\pi}{2} < x < \pi$ and $\frac{3\pi}{2} < x < 2\pi$

$x \in (\frac{\pi}{2} + k\pi, \pi + k\pi) \cup (\frac{3\pi}{2} + k\pi, 2\pi + k\pi)$

$k \in Z$

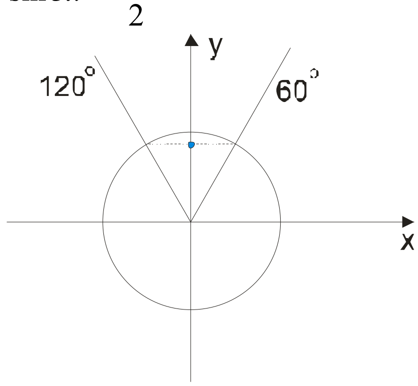
Examples:

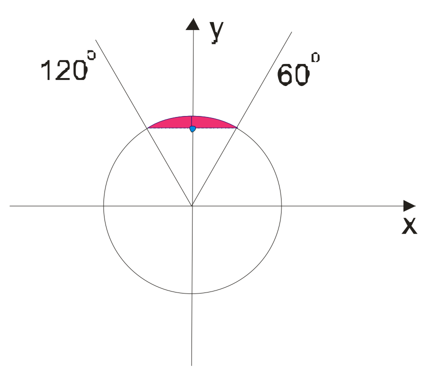
1) Solve the inequalities: $\sin 3x - \frac{\sqrt{3}}{2} \geq 0$

Solution:

$\sin 3x - \frac{\sqrt{3}}{2} = 0$

$\sin 3x = \frac{\sqrt{3}}{2}$





$\frac{\pi}{3} + 2k\pi \leq 3x \leq \frac{2\pi}{3} + 2k\pi$ and then $\frac{\pi}{9} + \frac{2k\pi}{3} \leq x \leq \frac{2\pi}{9} + \frac{2k\pi}{3}$

$k \in Z$

2) **Solve the inequalities:** $\sin x + \cos x < \sqrt{2}$

Solution:

This is the type of "support the introduction of argument"(see trigonometric equations)

a=1

$$tg\varphi = \frac{b}{a} \Rightarrow tg\varphi = \frac{1}{1} \Rightarrow tg\varphi = 1$$

b=1

$$\varphi = 45^o = \frac{\pi}{4}$$

c= $\sqrt{2}$

$$\frac{c}{\sqrt{a^2+b^2}} = \frac{\sqrt{2}}{\sqrt{1+1}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

So: $\sin(x+\varphi) = \frac{c}{\sqrt{a^2+b^2}} \Rightarrow \sin(x+\frac{\pi}{4}) = 1$

$$\sin(x+\frac{\pi}{4}) < 1$$

It does not answer only if $\sin(x+\frac{\pi}{4}) = 1$

$$x + \frac{\pi}{4} = \frac{\pi}{2} + 2k\pi$$

$$x = \frac{\pi}{2} - \frac{\pi}{4} + 2k\pi$$

$$x = \frac{\pi}{4} + 2k\pi$$

So, solution is $\forall x$ exsept $\frac{\pi}{4} + 2k\pi \longrightarrow x \neq \frac{\pi}{4} + 2k\pi, k \in Z$

3) **Solve the inequalities:** $2\sin^2 x + 5\sin x + 2 > 0$

Solution:

$2\sin^2 x + 5\sin x + 2 > 0 \rightarrow$ replacement sinx = t

$2t^2 + 5t + 2 > 0 \rightarrow$ see square inequalities!

$t_{1,2} = \dfrac{-5 \pm 3}{4}$

$t_1 = -\dfrac{1}{2}$

$t_2 = -2$

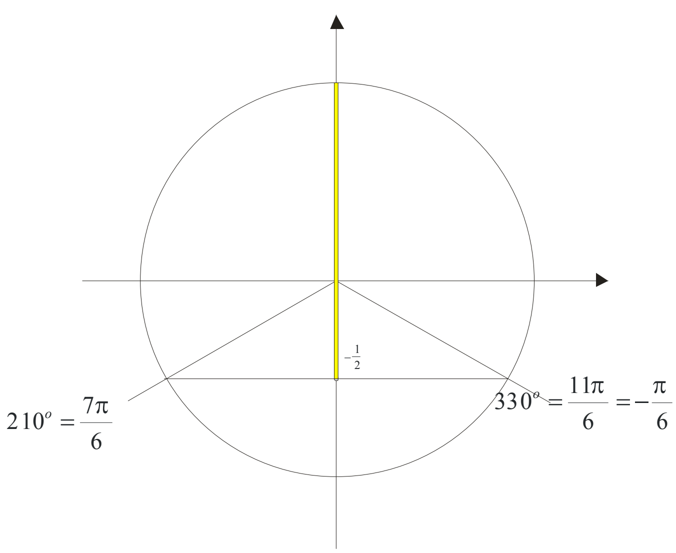
$\rightarrow$

$t \in (-\infty, -2) \cup (-\dfrac{1}{2}, \infty)$

$\sin x \in (-\infty, -2) \cup (-\dfrac{1}{2}, \infty)$

Since $-1 \leq \sin x \leq 1$ we have to make a correction of interval!

$\sin x \in \left(-\dfrac{1}{2}, 1\right] \longrightarrow \sin x > -\dfrac{1}{2}$



$-\dfrac{\pi}{6} + 2k\pi < x < \dfrac{7\pi}{6} + 2k\pi$ final solution!

$k \in Z$

4) Prove that applies to everyone $\alpha : \frac{1}{\sin^4\alpha} + \frac{1}{\cos^4\alpha} \geq 8$

Proof:

Transform expression on the left side!

$$\frac{1}{\sin^4\alpha} + \frac{1}{\cos^4\alpha} = \frac{\cos^4\alpha + \sin^4\alpha}{\sin^4\alpha + \cos^4\alpha} =$$

$$\sin^2\alpha + \cos^2\alpha = 1 /()^2$$

$$(\sin^2\alpha + \cos^2\alpha)^2 = 1$$

$$\sin^4\alpha + 2\sin^2\cos^2\alpha + \cos^4\alpha = 1 / add\frac{2}{2}$$

$$\sin^4\alpha + \cos^4\alpha = 1 - \frac{2\cdot 2\sin^2\alpha\cos^2\alpha}{2}$$

$$\sin^4\alpha + \cos^4\alpha = 1 - \frac{\sin^2 2\alpha}{2}$$

$$\sin^4\alpha + \cos^4\alpha = \frac{2 - \sin^2 2\alpha}{2} = \frac{1 + 1 - \sin^2 2\alpha}{2}$$

$$= \frac{1 + \cos^2 2\alpha}{2}$$

Let's go back to the task:

$$\frac{\cos^4\alpha + \sin^4\alpha}{\sin^4\alpha \cdot \cos^4\alpha} = \frac{1 + \cos^2 2\alpha}{2\sin^4\alpha \cdot \cos^4\alpha} = add(\frac{8}{8})$$

$$\frac{8(1 + \cos^2 2\alpha)}{16\sin^4\alpha\cos^4\alpha} = \frac{8(1 + \cos^2 2\alpha)}{\sin^4 2\alpha} \geq 8$$

And this certainly is!