# Trigonometric functions double angle

**Formulas are:**

1. $\sin 2\alpha = 2\sin\alpha\cos\alpha$

2. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

3. $tg2\alpha = \dfrac{2tg\alpha}{1-tg^2\alpha}$

4. $ctg2\alpha = \dfrac{ctg^2\alpha - 1}{2ctg\alpha}$

**Examples:**

1) **Prove :** a) $\sin 2\alpha = \dfrac{2tg\alpha}{1+tg^2\alpha}$

b) $\cos 2\alpha = \dfrac{1-tg^2\alpha}{1+tg^2\alpha}$

c) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

**Proofs:**

**a)** $\sin 2\alpha = \dfrac{2tg\alpha}{1+tg^2\alpha}$

$$\sin 2\alpha = \sin\alpha\cos\alpha =$$

$$\sin 2\alpha = \frac{\sin\alpha\cos\alpha}{1} = \frac{2\sin\alpha\cos\alpha}{\sin^2\alpha + \cos^2\alpha} = \text{( common and up and down} \cos^2\alpha \text{ )=}$$

$$\frac{\cancel{\cos^2\alpha}\cdot\dfrac{2\sin\alpha}{\cos\alpha}}{\cancel{\cos^2\alpha}\cdot\left(\dfrac{\sin^2\alpha}{\cos^2\alpha}+1\right)} = \frac{2tg\alpha}{tg^2\alpha+1} = \frac{2tg\alpha}{1+tg^2\alpha}$$

**b)**

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{\cos^2\alpha - \sin^2\alpha}{1} = \frac{\cos^2\alpha - \sin^2\alpha}{\sin^2\alpha + \cos^2\alpha} =$$

$$= \frac{\cos^2\alpha\left(1-\dfrac{\sin^2\alpha}{\cos^2\alpha}\right)}{\cos^2\alpha\left(\dfrac{\sin^2\alpha}{\cos^2\alpha}+1\right)} = \frac{1+tg^2\alpha}{tg^2\alpha+1} = \frac{1-tg^2\alpha}{1+tg^2\alpha},$$

**c)** $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$\sin 3\alpha = \sin(2\alpha + \alpha) \rightarrow$ Use the formula $\sin(2\alpha + \alpha)$
$= \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha \rightarrow$ Now the formula for double angle
$= (2\sin\alpha\cos\alpha)\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\cdot\sin\alpha$
$= 2\sin\alpha\cos^2\alpha + \sin\alpha\cos^2\alpha - \sin^3\alpha$
$= 3\sin\alpha\cos^2\alpha - \sin^3\alpha$
(from $\sin^2\alpha + \cos^2\alpha = 1$ is $\cos^2\alpha = 1 - \sin^2\alpha$ )
$= 3\sin\alpha(1 - \sin^2) - \sin^3\alpha$
$= 3\sin\alpha - 3\sin^3\alpha - \sin^3\alpha$
$= 3\sin\alpha - 4\sin^3\alpha$

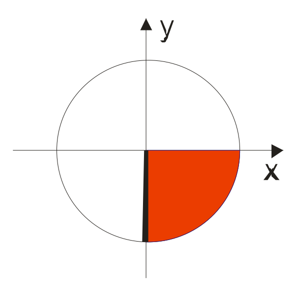
**2) We have:** $\cos\alpha = \frac{4}{5}$. **Find the value of double angles, if α is in the fourth quarter.**

**Solution:**

First, we calculate $\sin\alpha$ :

$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\sin^2\alpha = 1 - \cos^2\alpha$$
$$\sin^2\alpha = 1 - \left(\frac{4}{5}\right)^2$$
$$\sin^2\alpha = 1 - \frac{16}{25}$$
$$\sin^2\alpha = \frac{9}{25}$$
$$\sin\alpha = \pm\sqrt{\frac{9}{25}}$$

$\sin\alpha = \pm\frac{3}{5}$ ⟶ As the angle is from IV quarter . We'll take that $\sin\alpha = -\frac{3}{5}$

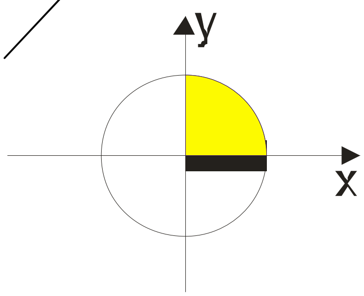



$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$= 2\left(-\frac{3}{5}\right)\cdot\frac{4}{5}$$
$$= -\frac{24}{25}$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= \left(\frac{4}{5}\right)^2 - \left(-\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$
$$tg 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

**3) We have $\sin\alpha$ =0,6 and $\alpha$ is in I quarter. Find the value of double angles.**

**Solution:**

$$\sin^2\alpha+\cos^2\alpha=1$$
$$\cos^2\alpha=1-\sin^2\alpha$$
$$\cos^2\alpha=1-(0,6)^2$$
$$\cos^2\alpha=1-0,36$$
$$\cos^2\alpha=0,64$$
$$\cos\alpha=\pm\sqrt{0,64}$$
$$\cos\alpha=\pm0,8$$
$$\cos\alpha=+0,8$$




$$\sin 2\alpha=2\sin\alpha\cos\alpha$$
$$=2\cdot0,6\cdot0,8$$
$$=2\cdot\frac{3}{5}\cdot\frac{4}{5}=\frac{24}{25}$$
$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$$
$$=\left(\frac{4}{5}\right)^2-\left(\frac{3}{5}\right)^2=\frac{7}{25}$$
$$tg2\alpha=\frac{\sin 2\alpha}{\cos 2\alpha}=\frac{\frac{24}{25}}{\frac{7}{25}}$$
$$tg2\alpha=\frac{24}{7}$$

**4) Prove:**

**a)** $\sin 15^o\cos 15^o=\frac{1}{4}$

**b)** $1-4\sin^2\alpha\cos^2\alpha=\cos^2 2\alpha$

**Solution:**

**a)**

$$\sin 15^o\cos 15^o=\{\text{add }\frac{2}{2}\}=\frac{2\sin 15^0\cos 15^0}{2}=\{\textbf{formula: }\sin 2\alpha=2\sin\alpha\cos\alpha\}=\frac{\sin 30^o}{2}=\frac{\frac{1}{2}}{2}=\frac{1}{4}$$

**b)**

$$1-4\sin^2\alpha\cos^2\alpha=\cos^2 2\alpha$$

$$1-4\sin^2\alpha\cos^2\alpha=\cos^2\alpha\ \{\textbf{formula }\sin 2\alpha=2\sin\alpha\cos\alpha,\ \textbf{it is: }4\sin^2\alpha\cos^2\alpha=\sin^2 2\alpha\}$$
$$=1-\sin^2 2\alpha=\cos^2 2\alpha$$

**5) Prove:**

**a)** $2\sin^2\alpha + \cos 2\alpha = 1$

$$2\sin^2\alpha + \cos 2\alpha = 2\sin^2\alpha + \cos^2\alpha - \sin^2\alpha$$
$$= \sin^2\alpha + \cos^2\alpha = 1$$

**b)** $\cos^4\alpha + \sin^4\alpha = 1 - 0{,}5\sin^2\alpha$

To prove this ,start from:

$$\sin^2\alpha + \cos^2\alpha = 1/ \text{ all the square}$$
$$\sin^4\alpha + 2\sin^2\alpha\cos^2\alpha + \cos^4\alpha = 1$$
$$\sin^4\alpha + \cos^4\alpha = 1 - \sin^2\alpha\cos^2\alpha$$
$$\sin^4\alpha + \cos^4\alpha = 1 - \frac{4\sin^2\alpha\cos^2\alpha}{2}$$
$$\sin^4\alpha + \cos^4\alpha = 1 - \frac{1}{2}\sin^2\alpha$$
$$\sin^4\alpha + \cos^4\alpha = 1 - 0{,}5\sin^2 2\alpha$$

**6) Prove identity**: $\cos 4\alpha + 4\cos 2\alpha + 3 = 8\cos^4\alpha$

**Proof:** We will go from left, to prove right side.

$$\cos 4\alpha + 4\cos 2\alpha + 3 =$$
$$\cos 2\cdot(2\alpha) + 4(\cos^2\alpha - \sin^2\alpha) + 3 =$$
$$\cos^2(2\alpha) - \sin^2(2\alpha) + 4\cos^2\alpha - 4\sin^2\alpha + 3 =$$
$$(\cos^2\alpha - \sin^2\alpha)^2 - (2\sin\alpha\cos\alpha)^2 + 4\cos^2\alpha - 4\sin^2\alpha + 3 =$$
$$(\cos^2\alpha - (1-\cos^2\alpha))^2 - 4\sin^2\alpha\cos^2\alpha + 4\cos^2\alpha - 4\sin^2\alpha + 3 = [\mathit{replace}\ \sin^2\alpha = 1 - \cos^2\alpha]$$
$$(2\cos^2\alpha - 1)^2 - 4\cos^2\alpha(1-\cos^2\alpha) + 4\cos^2\alpha - 4(1-\cos^2\alpha) + 3 =$$
$$4\cos^4\alpha \cancel{-4\cos^2\alpha} + 1 \cancel{-4\cos^2\alpha} + 4\cos^4\alpha \cancel{+4\cos^2\alpha} - 4 \cancel{+4\cos^2\alpha} + 3 =$$
$$= 8\cos^4\alpha$$

**7)** **If** $\sin\frac{x}{2}+\cos\frac{x}{2}=1{,}4$ **calculate** $\sin\alpha$

**Solution:**

$$\sin\frac{x}{2}+\cos\frac{x}{2}=1{,}4\;/()^2$$

$$\sin^2\frac{x}{2}+\underbrace{2\sin\frac{x}{2}\cos\frac{x}{2}}_{\sin\alpha}+\cos^2\frac{x}{2}=1{,}96$$

$$\underbrace{\sin^2\frac{x}{2}\;\ldots\;\cos^2\frac{x}{2}}_{1}$$

$$1+\sin\alpha=1{,}96$$
$$\sin\alpha=1{,}96-1$$
$$\sin\alpha=0{,}96$$

**8)** **Introduce** $tg3\alpha$ **as function of** $tg\alpha$

**Solution:**

$$tg3\alpha=tg(2\alpha+\alpha)=\frac{tg2\alpha+tg\alpha}{1-tg2\alpha\cdot tg\alpha}=$$

$$=\frac{\frac{2tg\alpha}{1-tg^2\alpha}+tg\alpha}{1-tg\alpha\cdot\frac{2tg\alpha}{1-tg^2\alpha}}=\frac{\frac{2tg\alpha+tg\alpha(1-tg^2\alpha)}{\cancel{1-tg^2\alpha}}}{\frac{1-tg^2\alpha+2tg^2\alpha}{\cancel{1-tg^2\alpha}}}$$

$$=\frac{2tg\alpha+tg\alpha-tg^3\alpha}{1+tg^2\alpha}=\frac{3tg\alpha-tg^3\alpha}{1+tg^2\alpha}$$

**9) Prove identity:** $\frac{1+\sin 2\alpha}{\sin\alpha+\cos\alpha}=\sqrt{2}\cos\left(\frac{\pi}{4}-\alpha\right)$

**Proof:**

$$\frac{1+\sin 2\alpha}{\sin\alpha+\cos\alpha}=\frac{\overbrace{\sin^2\alpha+\cos^2\alpha}^{1}+2\sin\alpha\cos\alpha}{\sin\alpha+\cos\alpha}=\frac{(\sin\alpha+\cos\alpha)^2}{\sin\alpha+\cos\alpha}$$

$$=\sin\alpha+\cos\alpha=(\text{in both addend will add }\frac{2}{2}=\frac{\sqrt{2}^2}{2})$$

$$\frac{\sqrt{2}^2}{2}\sin\alpha+\frac{\sqrt{2}^2}{2}\cos\alpha=\text{ (as a common }\sqrt{2}\text{ )}$$

$$\sqrt{2}\left(\frac{\sqrt{2}}{2}\sin\alpha+\frac{\sqrt{2}}{2}\cos\alpha\right)=(\text{because of: }\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}\text{ and }\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}\text{ })$$

$$\sqrt{2}\left(\sin\frac{\pi}{4}\sin\alpha+\cos\frac{\pi}{4}\cos\alpha\right)=$$

$$\sqrt{2}\left(\cos\frac{\pi}{4}\cos\alpha+\sin\alpha\sin\frac{\pi}{4}\right)=\{\text{This is a formula for }\cos(\alpha-\beta)\}$$

$$=\sqrt{2}\cos\left(\frac{\pi}{4}-\alpha\right)$$