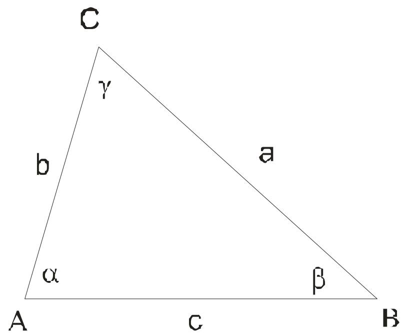
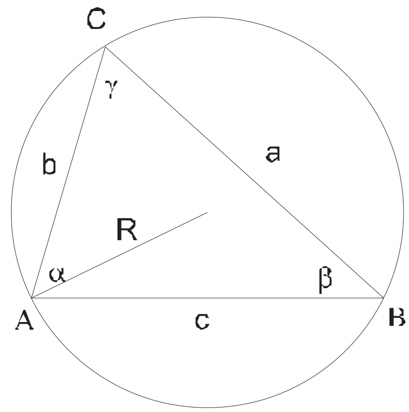
Sine and cosine theorem

Sine theorem:(law of sines)





$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

For a triangle with length of sides *a*, *b*, *c* and angles of α, β, γ respectively, the ratio of the length of side *a* to the sine of its corresponding angle α is equal to the ratio of the length of side *b* to the sine of its corresponding angle β.

Sine theorem applies:

a) **When we have two angle and length of one side**
b) **When we have length of two sides and angle opposed to one of the sides**

Cosine theorem:(law of cosines)

For a triangle with length of sides *a*, *b*, *c* and angles of α, β, γ respectively:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos je$$

Cosine theorem applies:

a) **When we have length of two sides and the angle between them**
b) **When we have length of all tree sides**

Some important "things" that derive from sinus and cosine theorem are:

Triangle area:

$$P = \frac{1}{2} a \sin\alpha$$

$$P = \frac{1}{2} b \sin\beta$$

$$P = \frac{1}{2} c \sin\gamma$$

$P = \frac{a \cdot b \cdot c}{4R}$, where is $s = \frac{a+b+c}{2}$, R is radius of circles described about the triangle

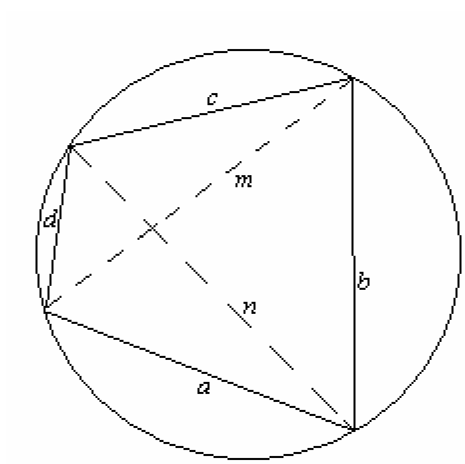
Median triangle:

$$t_a = \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2}$$

$$t_b = \frac{\sqrt{2c^2 + 2a^2 - b^2}}{2}$$

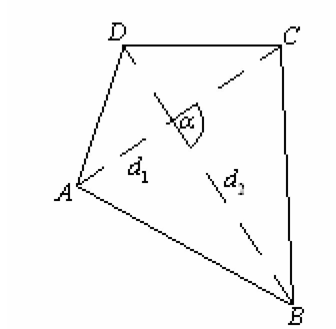
$$t_c = \frac{\sqrt{2a^2 + 2b^2 - c^2}}{2}$$

Ptolomerev theorem $m \cdot n = ac + bd$ **where is:**



Area of the quadrilateral is:

$$P = \frac{1}{2} d_1 \cdot d_2 \sin\alpha$$



EXAMPLES:

1) In the triangle ABC is $\alpha = 45^0$, $\beta = 60^0$ and the radius of the circle is $R = 2\sqrt{6}$.
Determine the other basic elements without use of table.

Solution:

$$\alpha = 45^o$$
$$\beta = 60^o$$
$$R = 2\sqrt{6}$$

First, we find angle γ

$$\alpha + \beta + je = 180^o$$
$$\gamma = 180^o - (45^o + 60^o)$$
$$\gamma = 75^o$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

$$\frac{a}{\sin\alpha} = 2R \Rightarrow \quad a = 2R\sin\alpha$$
$$a = 2\cdot 2\sqrt{6}\sin 45^o$$
$$a = 4\sqrt{6}\cdot\frac{\sqrt{2}}{\sqrt{2}} = 2\sqrt{12} = 4\sqrt{3}$$
$$a = 4\sqrt{3}$$

$$\frac{b}{\sin\beta} = 2R \Rightarrow \quad b = 2R\sin\beta$$
$$b = 2\cdot 2\sqrt{6}\sin 60^o$$
$$b = 4\sqrt{6}\cdot\frac{\sqrt{3}}{2} = 2\sqrt{18} = 6\sqrt{3}$$
$$b = 6\sqrt{3}$$

$$\frac{c}{\sin\gamma} = 2R \Rightarrow \quad c = 2R\sin\gamma$$
$$c = 2\cdot R\sqrt{6}\sin 75^o$$
$$c = 4\sqrt{6}\cdot\sin(45^o + 30^o)$$
$$c = 4\sqrt{6}\cdot(\sin 45^o\cos 30^o + \cos 45^o\sin 30^o)$$
$$c = 4\sqrt{6}\cdot\left(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right)$$
$$c = 2\left(3 + \sqrt{3}\right)$$

2) Determine length *b* in triangle ABC, if $a = 2\sqrt{3}, c = \sqrt{6}$ and angle $\beta = 105^0$

Solution:

$$\begin{aligned} a &= 2\sqrt{3}cm \\ c &= \sqrt{6}cm \\ \beta &= 105^o \\ \hline b &= ? \end{aligned}$$

Here we use cosine theorem!

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

Let's go first to find $\cos 105^o = ?$

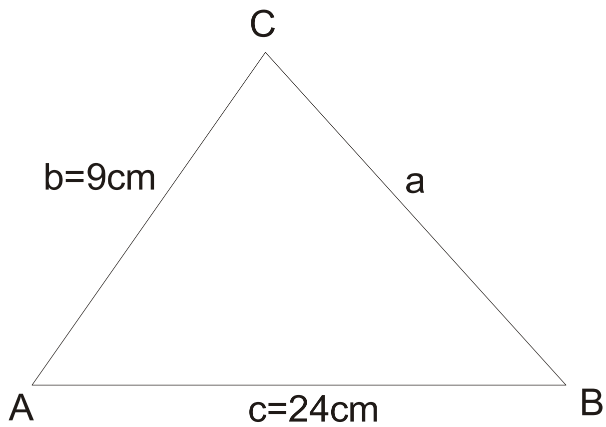
$$\begin{aligned} \cos 105^o &= \cos(60^o + 45^o) \\ &= \cos 60^o \cos 45^o - \sin 60^o \sin 45^o \\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}(1-\sqrt{3})}{4} \end{aligned}$$

$$\begin{aligned} b^2 &= (2\sqrt{3})^2 + (\sqrt{6})^2 - 2\cdot 2\sqrt{3}\cdot\sqrt{6}\cdot\frac{\sqrt{2}(1-\sqrt{3})}{4} \\ b^2 &= 12 + 6 - 6(1-\sqrt{3}) \\ b^2 &= 12 + 6 - 6 + 6\sqrt{3} \\ b^2 &= 12 + 6\sqrt{3} \\ b^2 &= (3+\sqrt{3})^2 \\ b &= 3 + \sqrt{3} \end{aligned}$$

$$\longrightarrow \quad \begin{aligned} 12 + 6\sqrt{3} &= (3+\sqrt{3})^2 \\ &= 3^2 + 2\cdot 3\sqrt{3} + \sqrt{3}^2 \\ &= 9 + 6\sqrt{3} + 3 \\ &= 12 + 6\sqrt{3} \end{aligned}$$

3) In the triangle ABC is AB = 24cm, AC = 9cm and angle $\alpha = 60^0$. Find BC =? and R=?

Solution:



$$b = 9cm$$
$$c = 24cm$$
$$\alpha = 60^o$$
$$\overline{}$$
$$a = ?, R = ?$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$a^2 = 9^2 + 24^2 - 2\cdot 9\cdot 24\cdot\cos 60^o$$
$$a^2 = 81 + 576 - 2\cdot 9\cdot 24\cdot\frac{1}{2}$$
$$a^2 = 441$$
$$a = \sqrt{441}$$
$$a = 21cm$$

$$\frac{a}{\sin\alpha} = 2R \Rightarrow \frac{21}{\sin 60^o} = 2R$$
$$\frac{21}{\frac{\sqrt{3}}{2}} = 2R$$
$$2R = \frac{42}{\sqrt{3}}$$
$$R = \frac{21}{\sqrt{3}}$$
$$R = \frac{21}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$
$$R = \frac{21\sqrt{3}}{\sqrt{3}}$$
$$R = 7\sqrt{3}cm$$

4) In the triangle ABC is:

$$a - b = 3$$
$$\gamma = 60^o$$
$$R = \frac{7\sqrt{3}}{3}$$
$$\overline{\qquad\qquad}$$
$$a, b, c = ?$$

Solution:

$$\frac{c}{\sin\gamma} = 2R \Rightarrow \quad c = 2R\sin\gamma$$
$$c = 2 \cdot \frac{7\sqrt{3}}{3} \cdot \sin 60^o$$
$$c = 2 \cdot \frac{7\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{2}$$
$$c = 7$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$7^2 = (b+3)^2 + b^2 - 2b(b+3)\cos 60^o$$
$$49 = (b+3)^2 + b^2 - 2b(b+3) \cdot \frac{1}{2}$$
$$49 = b^2 + 6b + 9 + b^2 - b^2 - 3b$$

$b^2 + 3b - 40 = 0 \rightarrow$ quadratic equation by "b"

$$b_{1,2} = \frac{-3 \pm 13}{2}$$
$$b_1 = 5$$

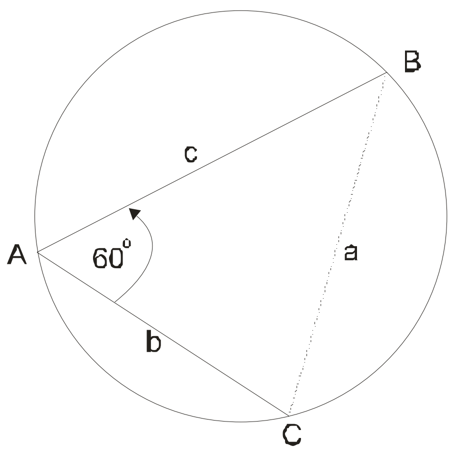
$b_2 = -8 \rightarrow$ this is not a solution because the length of the page can not be a negative number

So: $\quad b = 5$

$$a = b + 3$$
$$a = 5 + 3$$
$$a = 8$$

5) **In circle we have AB=8cm and AC=5cm, and angle between them** $\alpha = 60^0$ **.Find R=?**

Solution:



$$b = 5cm$$
$$c = 8cm$$
$$\alpha = 60^o$$

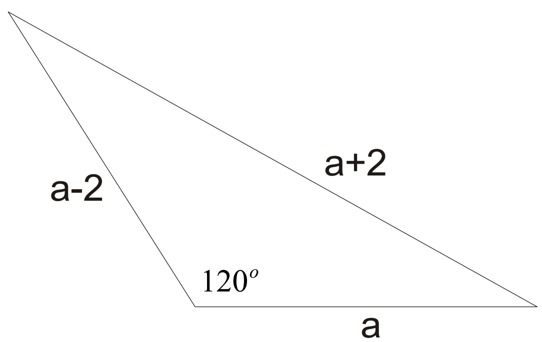
$$R = ?$$

$$a = b^2 + c^2 - 2bc\cos\alpha$$
$$a^2 = 5^2 + 8^2 - 2\cdot 5\cdot 8\cdot \cos 60^o$$
$$a^2 = 25 + 64 - 2\cdot 40\cdot \frac{1}{2}$$
$$a^2 = 89 - 40$$
$$a^2 = 49$$
$$a = 7cm$$
$$\frac{a}{\sin\alpha} = 2R \Rightarrow$$

$$\frac{7}{\sin 60^o} = 2R$$
$$2R = \frac{7}{\frac{\sqrt{3}}{2}}$$
$$R = \frac{7}{\sqrt{3}}$$
$$R = \frac{7}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$
$$R = \frac{7\sqrt{3}}{3}$$

6) If the page length of the triangle are $a-2, a, a+2$, and angle 120^0 ,find a=?, b=? and c=?

Solution:



$a = a$

$b = a - 2$

$c = a + 2$

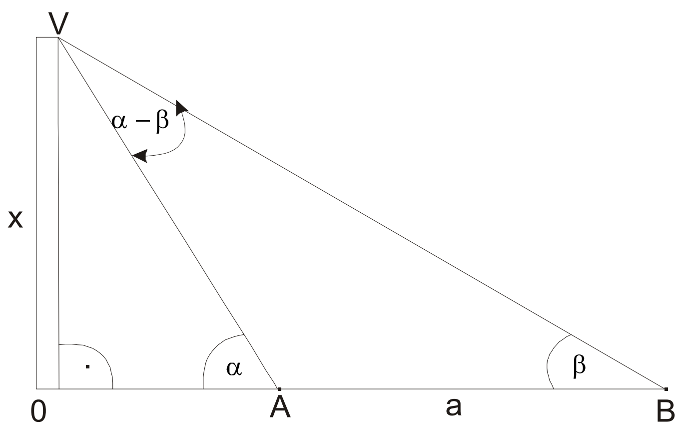
$$c^2 = a^2 + b^2 - 2ab\cos je$$
$$(a+2)^2 = a^2 + (a-2)^2 - 2a(a-2)\cos 120^o$$
$$(a+2)^2 = a^2 + (a-2)^2 - 2a(a-2)\cdot\left(-\frac{1}{2}\right)$$
$$a+2)^2 = a^2 + (a-2)^2 + a(a-2)$$
$$a^2 + 4a + 4 = a^2 + a^2 - 4a + 4 + a^2 - 2a$$
$$0 = 2a^2 - 10a$$
$$2a(a-5) = 0$$
$$a = 0 \rightarrow \text{impossible}$$

$$a = 5$$
$$b = a - 2 = 5 - 2 = 3$$
$$b = 3$$
$$c = a + 2$$
$$c = 7$$

7) Calculate the amount of factory chimney, which is located on the horizontal soil, if the top of the chimney we can see under angle α from the point A , from point B under angle β. Points A and B also belong to horizontal plane and their distance is AB= a.

Solution:

Here is the most important to draft problem!



With OV=X mark required height

First, find unknown corners $\angle OVA$ and $\angle AVB$

$$\left.\begin{aligned}\angle OVA &= 90^o - \alpha \\ \angle OVB &= 90^o - \beta\end{aligned}\right\} \Rightarrow \begin{aligned} & AVB = \angle OVB - \angle OVA \\ & = (90^o - \beta) - (90^o - \alpha) \\ & = 90^o - \beta - 90^o + \alpha \\ & \angle AVB = \alpha - \beta \end{aligned}$$

Primenimo sinusnu teoremu na trougao ABV

$$\frac{a}{\sin(\alpha-\beta)} = \frac{AV}{\sin\beta} \Rightarrow AV = \frac{a\sin\beta}{\sin(\alpha-\beta)} \quad \text{next:}$$

$$\sin\alpha = \frac{x}{AV} \Rightarrow \quad x = AV(\sin\alpha)$$

$$x = \frac{a\sin\beta\sin\alpha}{\sin(\alpha-\beta)}$$

$$x = \frac{a\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$$

8) In the triangle ABC is $a-b=1,\ h_c=\frac{3}{2}$, R=4. Without the use of table calculate angle α.

Solution:

$$a-b=1$$
$$h_c=\frac{3}{2}$$
$$R=4$$
$$\alpha=?$$

First, we use forms for the area of a triangle:

$$P=\frac{c\cdot h_c}{2},\quad P=\frac{abc}{4R}$$

So:

$$\frac{c\cdot h_c}{2}=\frac{abc}{4R}$$
$$2ab=4Rh_c$$
$$ab=2Rh_c$$
$$ab=2\cdot 4\cdot h_c$$
$$ab=2\cdot 4\cdot\frac{3}{2}$$
$$ab=12$$

Now a system:

$$a-b=1$$
$$ab=12$$
$$a=b+1$$
$$b(b+1)=12$$
$$b^2+b-12=0$$
$$b_{1,2}=\frac{-1\pm 7}{2}$$
$$b_1=3$$
$$b_2=-4 \qquad \text{impossible}$$

So: $b=3\Rightarrow a=3+1=4\Rightarrow a=4$

$$\frac{a}{\sin\alpha} = 2R \Rightarrow \sin\alpha = \frac{a}{2R}$$

$$\sin\alpha = \frac{4}{8}$$

$$\sin\alpha = \frac{1}{2}$$

$\alpha = 30^o$ because $\sin = 30^o = \frac{1}{2}$

$\alpha = 30^o$ **is solution!**

9) Calculate the length of a,b and c triangle if $P = 3\sqrt{3}$, $\alpha = 60^0$, b+c=7 .

Solution:

$P = 3\sqrt{3}$
$\alpha = 60^o$
$b + c = 7$
$a, b, c = ?$

$$P = \frac{1}{2}bc\sin\alpha$$

$$3\sqrt{3} = \frac{1}{2}bc\sin 60^o$$

$$3\sqrt{3} = \frac{1}{2}bc\cdot\frac{\sqrt{3}}{3}$$

$$bc = 12$$

$$b + c = 7$$
$$bc = 12$$

c=7-b ⟶ bc=12

$$b\cdot(7-b) = 12$$
$$7b - b^2 = 12$$
$$b^2 - 7b + 12 = 0$$
$$b_{1,2} = \frac{7\pm 1}{2}$$
$$b_1 = 4 \Rightarrow c = 3$$
$$b_2 = 3 \Rightarrow c = 4$$

So, we have two options:

$b_1 = 4, c = 3$ or $b_2 = 3, c = 4$

Use now cosine theorem:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$a^2 = 4^2 + 3^2 - 2\cdot 4\cdot 3\cdot \cos 60^o$$

$$a^2 = 16 + 9 - 2\cdot 12\cdot \frac{1}{2}$$

$$a^2 = 25 - 12$$

$$a^2 = 13$$

$$a = \sqrt{13}$$